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ON AN ERROR TERM INVOLVING THE TOTIENT FUNCTION

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This paper provides a mean-square result on the error term $E(t)$ in the asymptotic formula

$$\sum_{n \leqslant t} \frac{n}{\phi(n)} = \frac{315}{2\pi^4} \zeta(3)\, t - \tfrac{1}{2} \log t - c_1 + E(t)$$

where $\phi(n)$ is Euler's totient function and c_1 an effective constant. It is proven that

$$\int_0^x (E(t))^2\, dt = Cx + O\left(x^{4/5} (\log x)^{3/5} (\log \log x)^{6/5}\right)$$

with $C \approx 0{,}546$.

1. INTRODUCTION

For a positive integer n, let as usual $\phi(n)$ denote the number of positive integers $m \leqq n$ which are coprime to n. The rough observation that '$\phi(n)$ is on average of the order n' was supported in classic times by the asymptotics

$$\sum_{n \leqslant x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x),\ H(x) = O(\log x) \qquad \ldots(1)$$

and

$$\sum_{n \leqslant x} \frac{n}{\phi(n)} = \frac{315}{2\pi^4} \zeta(3)\, x + E^*(x),\ E^*(x) = O(\log x) \qquad \ldots(2)$$

see e. g. Landau[2] and Pillai and Chowla[4]. The error term $H(x)$ was subject of a detailed study in the sequel: It was shown that $H(x) = O((\log x)^{2/3} (\log \log x)^{4/3})$ (see Walfisz[11], p. 114), and that $H(x)$ has a significant oscillation behaviour, displayed by the mean-value results

$$\int_0^x H(t)\, dt = O\left(x \exp\left(-c (\log x)^{3/5} (\log \log x)^{-1/5}\right)\right),\ c > 0 \qquad \ldots(3)$$

(see Suryanarayana and Sitaramachandrarao[8,9]) and

$$\int_0^x H^2(t)\,dt \sim \frac{1}{2\pi^2}\,x \qquad ...(4)$$

(see Chowla[1] and, more recently, Pétermann[3] for a quantitative version).

A thorough investigation of $E^*(x)$ was initiated only a few years ago by Sitaramachandrarao[6,7]. He proved that

$$E^*(x) = -\tfrac{1}{2}\log x - \tfrac{1}{2}\left(\gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}\right) + E(x) \qquad ...(5)$$

(here γ denotes the Euler-Mascheroni constant and the sum extends over all primes p), where $E(x) = O((\log x)^{2/3})$ and

$$\int_0^x E(t)\,dt = O(x^{4/5}). \qquad ...(6)$$

In the present note we study the asymptotic behaviour of the mean-square of $E(t)$, showing that

$$\int_0^x E^2(t)\,dt \sim Cx$$

(with a constant $C > 0$) which, together with (6), sheds some light on the oscillating nature of $E(t)$. More precisely, we prove the following.

Throrem—Let $E(t)$ be defined by (2) and (5), then we have the asymptotic

$$\int_0^x E^2(t)\,dt = Cx + O(x^{4/5}(\log x)^{3/5}(\log\log x)^{6/5}). \qquad ...(7)$$

where

$$C = \frac{1}{12}\prod_p \left(1 + \left(\frac{3}{p^2} - \frac{2}{p^3}\right)\left(1 - \frac{1}{p}\right)^{-2}\right) \approx 0{,}546. \qquad ...(8)$$

2. Some Technical Tools (see Walfisz[10])

Lemma A—For a large real parameter y,

$$\sum_{1\leq m,n\leq y} \sum_{\substack{u,v=1\\ um\neq vn}}^{\infty} (uv)^{-1}\,|\,um - vn\,|^{-1} = O(y\log y)$$

$$\sum_{1\leq m,n\leq y} \sum_{u,v=1}^{\infty} (uv)^{-1}(um + vn)^{-1} = O(y).$$

Lemma B—For arbitrary positive integers m, n

$$\sum_{\substack{u,v=1 \\ um=vn}}^{\infty} (uv)^{-1} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn}$$

where $(.\,,\,.)$ denotes the greatest common divisor.

Lemma C (cf. Chowla[1])—Let c_n denote some arithmetic function satisfying $c_n = O(n^{\epsilon})$ for every $\epsilon > 0$, and put $g_n = \sum_{d|n} c_d$. Then we have

$$\sum_{u,v=1}^{\infty} c_u\, c_v\, (u, v)^2 (uv)^{-2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} g_n^2\, n^{-2}.$$

3. Proof of the Theorem

For $t \geqq 3$, we put $y(t) = t^{4/5} (\log t)^{-2/5} (\log \log t)^{-4/5}$, then it has been proved in Sitaramachandrarao[7], formula (4.5), that (in our notation)

$$E(t) = -S(t) + O(t^{-1/5} (\log t)^{3/5} (\log \log t)^{6/5}) \qquad \ldots(9)$$

where

$$S(t) := \sum_{1 \leqq n \leqq y(t)} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{t}{n}\right)$$

μ is the Moebius function and $P(w) = w - [w] - \frac{1}{2}$. (Properly speaking, this result was given in Sitaramachandrarao[8] for $\bar{y}(t) = t\,[t^{1/5} (\log t)^{2/5} (\log \log t)^{4/5}]^{-1}$ instead of $y(t)$, but we easily infer from $(\phi(n))^{-1} = O(n^{-1} \log \log n)$ (see e. g. Prachar[5], p. 24) that

$$\sum_{y(t) < n \leqq \bar{y}(t)} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{t}{n}\right) = O(t^{-1/5}).$$

Now choose a constant a such that $y(t)$ increases for $t \geqq a$ and let z denote the inverse function of y; furthermore, put $M = M(m, n) = \max\{z(m), z(n), a\}$. Then we have for $x > a$

$$Q_0(x) := \int_a^x S^2(t)\, dt = \int_a^x \sum_{m,n \leqq y(t)} \frac{\mu^2(m)\, \mu^2(n)}{\phi(m)\, \phi(n)} P\left(\frac{t}{m}\right) P\left(\frac{t}{n}\right) dt$$

$$= \sum_{m,n < y(x)} \frac{\mu^2(m)\, \mu^2(n)}{\phi(m)\, \phi(n)} I(m, n) \qquad \ldots(10)$$

where

$$I(m,n) = \int_M^x P\left(\frac{t}{m}\right) P\left(\frac{t}{n}\right) dt.$$

By classic results on Fourier series,

$$I(m,n) = \frac{1}{\pi^2} \sum_{u,v=1}^{\infty} (uv)^{-1} \int_M^x \sin\left(2\pi u \frac{t}{n}\right) \sin\left(2\pi v \frac{t}{m}\right) dt$$

$$= \frac{1}{2\pi^2} \sum_{u,v=1}^{\infty} (uv)^{-1} \left(\int_M^x \cos\left(2\pi t \left(\frac{u}{n} - \frac{v}{m}\right)\right) dt \right.$$

$$\left. - \int_M^x \cos\left(2\pi t\left(\frac{u}{n} + \frac{v}{m}\right)\right) dt \right) = \frac{1}{2\pi^2} \sum_{\substack{u,v=1 \\ um=vn}}^{\infty} (uv)^{-1} (x - M)$$

$$+ O\left(\sum_{\substack{u,v=1 \\ um \neq vn}}^{\infty} (uv)^{-1} \left| \frac{u}{n} - \frac{v}{m} \right|^{-1} + \sum_{u,v=1}^{\infty} (uv)^{-1} \right.$$

$$\left. \times \left(\frac{u}{n} + \frac{v}{m} \right)^{-1} \right). \qquad \ldots(11)$$

Using again that $(\phi(n))^{-1} = O(n^{-1} \log\log n)$ and appealing to Lemma A, we thus get

$$Q_0(x) = \frac{1}{2\pi^2} \sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} (x - M(m,n)) \sum_{\substack{u,v=1 \\ um=vn}}^{\infty} (uv)^{-1}$$

$$+ O(y(x) \log x (\log\log x)^2).$$

By Lemma B,

$$Q_0(x) = \frac{1}{12} \sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} (x - M(m,n)) \frac{(m,n)^2}{mn}$$

$$+ O(y(x) \log x (\log\log x)^2). \qquad \ldots(12)$$

Furthermore, we easily deduce the estimates

$$\sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} M(m,n) \frac{(m,n)^2}{mn} \ll \log\log x)^2 \sum_{m \leqq n \leqq y(x)}$$

(*equation continued on p. 541*)

$$\times (m, n)^2 m^{-2} n^{-3/4+\epsilon} \ll (\log\log x)^2 \sum_{d \leq y(x)} d^{-3/4+\epsilon}$$

$$\sum_{m \leq n \leq \frac{y(x)}{d}} m^{-2} n^{-3/4+\epsilon} \ll (y(x))^{1/4+2\epsilon} \qquad \ldots(13)$$

and

$$x \sum_{\max(m,n) > y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} \frac{(m, n)^2}{mn} \ll x \sum_{\max(m,n) > y(x)}$$

$$\times (m, n)^2 (m\,n)^{-7/4} \ll x \sum_{d=1}^{\infty} d^{-3/2} \sum_{m > y/d} m^{-7/4}$$

$$\ll x \sum_{d=1}^{\infty} d^{-3/2} \min\left(\left(\frac{y}{d}\right)^{-3/4}, 1\right) \ll x\,(y(x))^{-1/2}$$

$$\ll x^{3/5+\epsilon}. \qquad \ldots(14)$$

Thus (12) may be simplified to

$$Q_0(x) = Cx + O(y(x) \log x (\log\log x)^2) \qquad \ldots(15)$$

where

$$C = \frac{1}{12} \sum_{m,n=1}^{\infty} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} \frac{(m, n)^2}{m\,n}.$$

Going back to (9) and applying the Cauchy-Schwarz inequality for integrals, we obtain

$$\int_0^x E^2(t)\,dt = C\,x + O(x^{4/5} (\log x)^{3/5} (\log\log x)^{6/5})$$

which is just the assertion of our theorem, apart from the representation of the constant C. To evalute the latter, we apply Lemma C with $c_n = \mu^2(n) \dfrac{n}{\phi(n)}$.
By multiplicativity,

$$g_n = \sum_{m|n} \mu^2(m) \frac{m}{\phi(m)} = \prod_{p|n} \frac{2p-1}{p-1}$$

and

$$C = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} g_n^2 n^{-2} = \frac{1}{2\pi^2} \prod_p \left(1 + \left(\frac{2p-1}{p-1}\right)^2 \sum_{k=1}^{\infty} p^{-2k}\right)$$

(equation continued on p. 542)

$$= \frac{1}{2\pi^2} \prod_p \frac{p^4 - 2p^3 + 4p^2 - 2p}{(p-1)^3 (p+1)} = \frac{\zeta(2)}{2\pi^2} \prod_p \left(1 - \frac{2}{p} + \frac{4}{p^2} - \frac{2}{p^3}\right)\left(1 - \frac{1}{p}\right)^{-2} = \frac{1}{12} \prod_p \left(1 + \left(\frac{3}{p^2} - \frac{2}{p^3}\right) \times \left(1 - \frac{1}{p}\right)^{-2}\right)$$

which completes the proof of our theorem.

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A FIXED POINT THEOREM FOR GENERALIZED CONTRACTION MAP

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AND

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In this paper we prove a fixed point theorem for a generalized contraction map introduced by Altman and then derive a few known results as corollaries.

Altman[2] proved the following interesting theorem : Let x be a complete metric space and $f : x \to x$ a generalized contraction, i.e.,

$$d(fx, fy) \leqslant Q(d(x, y)) \text{ for all } x, y \in X,$$

where Q satisfies the following :

(a) $0 < Q(t) < t$, for all $t \in (0, t_1]$,

(b) $g(t) = t/(t - Q(t))$ is nonincreasing,

(c) $\int_0^{t_1} g(t)\, dt < \infty$

and

(d) Q is nondecreasing.

Then f has a unique fixed point (see also Altman[1]).

Recently Watson *et al.*[6] pointed out that the fixed point is not necessarily unique under conditions (a), (b) (c) and (d). Carbone and Singh[3] gave a suitable example showing that the fixed point is, indeed, not unique.

Watson *et al.*[6] proved a theorem for a pair of mappings showing that $Fx = Gx$ has a unique solution under a set of conditions, where F is a generalized contraction

and G is an expansive map. Their theorem improves a result due to Norris and Sehgal[4].

Our aim is to prove the following theorem and to derive a few known results as corollaries.

Theorem 1—Let X be a complete metric space and let $f, h : X \to X$ be continuous functions such that

$$d(hx, hy) \leqslant Q(m(x, y)) \text{ for } x, y, \in X$$

where

$$m(x, y) = \max\left\{d(fx, fy), d(fx, hx), d(fy, hy), \frac{d(fx,hy) + d(fy,hx)}{2}\right\}.$$

Also suppose

(i) f and h are weakly commuting, i.e.

$$d(hfx, fhx) \leqslant d(fx, hx), \text{ and}$$

(ii) $h(X) \subset f(X)$.

Then f and h have a unique common fixed point. (i.e., there exists $x_0 \in X$ such that $fx_0 = x_0 = hx_0$).

In this case Q satisfies the following :

Q is a real-valued function such that

(a) $0 < Q(y) < y$ for $y > 0$, and $Q(0) = 0$,

(b) $g(y) = y/(y - Q(y))$ is nonincreasing on $(0, \infty)$,

(c) $\int_0^{y_1} g(y)\, dy < \infty$ for each $y_1 > 0$,

and

(d) $Q(y)$ is nondecreasing.

Proof : Suppose x and y are distinct common fixed points of f and h. Then $m(x, y) > 0$, since $fx \neq hy$. Hence,

$$d(hx, hy) \leqslant Q(m(x, y))$$

$$< \max\{d(fx, fy), 0, 0, d(fx, fy)\},$$

a contradiction.

To prove the existence, take x_0 in X and set $t_1 = d(hx_0, fx_0)$. Suppose $t_1 = 0$. Then

$$d(hhx_0, hx_0) \leqslant Q(m(hx_0, x_0))$$

where

$$m(hx_0, x_0) = \max\left\{d(fhx_0, fx_0), d(fhx_0, hhx_0), d(fx_0, hx_0), \frac{d(fhx_0, hx_0) + d(fx_0, hhx_0)}{2}\right\}.$$

Since f and h are weakly commuting and $fx_0 = hx_0$,

we have

$$d(fhx_0, hhx_0) = 0.$$

Hence

$$m(hx_0, x_0) = d(hhx_0, hx_0).$$

Note that $m(hx_0, x_0)$ must be zero, otherwise $m(hx_0, x_0) > 0$ would imply

$$d(hhx_0, hx_0) \leqslant Q(m(hx_0, x_0)) < d(hhx_0, hx_0)$$

a contradiction.

Thus $m(hx_0, x_0) = 0$, i.e., hx_0 is a fixed point of h.

But then

$$ffx_0 = fhx_0 = hhx_0 = hx_0 = fx_0$$

i.e.,

$$fx_0 = hx_0 \text{ is a fixed point of } f.$$

We may assume, now that $t_1 > 0$. Since $h(X) \subset f(X)$ there exists an $x_1 \in X$ with $fx_1 = hx_0$. In general, define $\{x_n\} \subset X$ so that $fx_n = hx_{n-1}, n \geqslant 1$.

Without loss of generality we may assume that $fx_n \neq hx_n$ for each n. For if $fx_n = hx_n$ for some n, then a repeat of the above argument, with x_0 replaced by x_n, yields fx_n as a common fixed point of f and h.

Define $\{t_n\}$ by $t_{n+1} = Q(t_n)$, with $t_1 = d(hx_0, fx_0)$. It then follows by assumption a) of Theorem 1 that

(i) $0 < t_{n+1} \leqslant t_n \leqslant t_1, n \geqslant 1$. Moreover, by hypotheses (b) and (c), the series $\sum_{n \geqslant 1} t_n$ converges (see Altman[1]). Furthermore, by induction on $n \in N$, we have

(ii) $d(hx_n, hx_{n-1}) \leqslant t_{n+1}, n \geqslant 1$.

Indeed, for $n = 1$,

$$d(hx_1, hx_0) \leqslant Q(m(x_1, x_0))$$

where

$$m(x_1, x_0) = \max\left\{d(fx_1, fx_0), d(fx_1, hx_1), d(fx_0, hx_0), \frac{d(fx_1, hx_0) + d(fx_0, hx_1)}{2}\right\}$$

$$= \max\left\{d(hx_0, fx_0), d(hx_0, hx_1), d(fx_0, hx_0), \frac{d(fx_0, hx_1)}{2}\right\}$$

$$= \max\{d(hx_0, fx_0), d(hx_0, hx_1)\} > 0.$$

Now, if $m(x_1, x_0) = d(hx_0, hx_1)$, then

$$d(hx_1, hx_0) \leqslant Q(m(x_1, x_0)) < d(hx_0, hx_1)$$

a contradiction.

Then

$$m(x_1, x_0) = d(hx_0, fx_0) = t_1.$$

Thus (ii) is proved for $n = 1$.

Assume now that (ii) holds for some $n > 1$. Then

$$d(hx_{n+1}, hx_n) \leqslant Q(m(x_{n+1}, x_n)),$$

where

$$m(x_{n+1}, x_n) = \max\left\{d(fx_{n+1}, fx_n), d(fx_{n+1}, hx_{n+1}), d(fx_n, hx_n), \frac{d(fx_{n+1}, hx_n) + d(fx_n, hx_{n-1})}{2}\right\}$$

$$= \max\{d(hx_{n+1}, hx_n), d(hx_n, hx_{n-1})\}.$$

Note that by the assumption $fx_n \neq hx_n$ for all n, $m(x_{n+1}, x_n) > 0$ for all n. If $m(x_{n+1}, x_n) = d(hx_{n+1}, hx_n)$, then we get

$$d(hx_{n+1}, hx_n) \leqslant Q(m(x_{n+1}, x_n)) < d(hx_{n+1}, hx_n), \text{ a contradiction.}$$

Therefore,

$$m(x_{n+1}, x_n) = d(hx_n, hx_{n-1})$$

and

$$d(hx_{n+1}, hx_n) \leqslant Q(d(hx_n, hx_{n-1}))$$
$$\leqslant Q(t_{n+1}) = t_{n+2}.$$

Clearly $\{hx_n\}$ is a Cauchy sequence. In fact, if m and n are natural numbers with $m \leqslant n$, then

$$d(hx_m, hx_n) \leqslant \sum_{i=m}^{n-1} d(hx_i, hx_{i+1}) < \sum_{i=m}^{n-1} t_{i+2}.$$

The convergence of $\sum_{n \geqslant 1} t_n$ implies that $\{hx_n\}$ is a Cauchy sequence, hence converges to a point $y \in X$. Since $hx_n = fx_{n+1}$, $\{fx_n\}$ also converges to y. Since f is continuous we get $fhx_n \to fy$. But f and h weakly commute. Hence we get $d(hfx_n, fy) \leqslant d(hfx_n, fhx_n) + d(fhx_n, fy)$, and $hfx_n \to fy$.

Since h is also continuous, $hfx_n \to hy$, so $hy = fy$.

Then, a repeat of the argument at the beginning of the proof with x_0 replaced by y, yields $hy = fy$ as a common fixed point of f and h.

The following results follow as Corollaries :

Corollary 1—If we replace weakly commuting by the commuting property i.e. $fhx = hfx$ for all $x \in X$, in Theorem 1, then f and h have a unique common fixed point. Note : Recall that commuting maps are weakly commuting, but not conversely (see Sessa[5]).

Corollary 2—If $m(x, y)$ is replaced by $d(fx, fy)$ in Theorem 1, then f and h have a unique common fixed point.

Corollary 3—We get a result due to Carbone and Singh[3] by putting $d(fx, fy)$ for $m(x, y)$ and commuting for weakly commuting in Theorem 1.

Corollary 4—In Corollary 3, if we put $f = I$, the identity function, then we get a theorem of Watson *et al.*[6].

Theorem 1 can be used to find the solution of an operator equation of the form $hx = Gx$, under suitable conditions on G.

We state the following given in Watson *et al.*[6].

Theorem 2—Let $h, G : X \to X$ be such that

(i) h is as in Theorem 1 with $f = I$, and $m(x, y) = d(fx, fy)$,

(ii) $d(Gx, Gy) \geqslant d(x, y)$ for all $x, y \in X$ and

(iii) $h(X) \subseteq G(X)$.

Then $hx = Gx$ has a unique solution z and for every

$$x_0 \in X, \lim_{n \to \infty} (G^{-1} h)^n x_0 = z.$$

In this case $G^{-1} h$ satisfies the conditions of Corollary 4 (see Watson *et al.*[6] for details).

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SEQUENCES OF MAPPINGS CONVERGING TO A CONTRACTION MAPPING

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We consider sequences of mappings $\{T_n\}$ from a metric space X into itself, which converge to a contraction mapping T. We define $T^{n+1} = T_{n+1} T^n$, $T^1 = T_1$ and we investigate the convergence of the sequence $\{T^n x\}$ for any point x in X.

1. INTRODUCTION

Nadler[1] investigated metric spaces (X, d) and sequences of functions $\{T_n\}$, $T_n: X \to X$, which converge on X to a contraction mapping T with fixed point a. He supposed that the functions $\{T_n\}$ have at least one fixed point for each $n = 1, 2, 3, \ldots$ and proved the convergence of the sequence of these fixed points to the fixed point a of T.

In this article we consider such sequences of functions, we define $T^{n+1} = T_{n+1}T^nT^1 = T_1$ and study the convergence of the sequence $\{T^n x\}$ for a point x in X.

2. CONVERGENCE OF THE SEQUENCE IN A METRIC SPACE

If $\{T_n\}$ is a sequence of mappings from a space X into itself then for every positive integer n, we define $T^{n+1} = T_{n+1} T^n$ with $T^1 = T_1$. If $n > m$ and n is a positive integer, by T_m^n we mean: $T_m^n = T_n T_{n-1} T_{n-2} \ldots T_m$.

Theorem 1—Let (X, d) be a metric space, let $\{T_n\}$ be a sequence of functions of X into itself and let $T: X \to X$ be a contraction mapping with fixed point a. If the sequence $\{T_n\}$ converges uniformly to T, then the sequence $\{T^n x\}$ converges to a for all $x \in X$.

PROOF : The proof is given in three steps.

Suppose k is the Lipschitz constant of T and choose $k_1 \in \mathrm{R}$ such that $0 < k < k_1 < 1$.

Step I—We shall show that for any $\epsilon > 0$ there exists an $m_0 \in \mathbb{N}$ such that $n \geqslant m_0$ gives :

$$d(T^n a, T^n x) < \epsilon \qquad \ldots(2.1)$$

for every $x \in X$.

Let $\epsilon > 0$ be given. Then for $\epsilon' = (k_1 - k)\epsilon > 0$ there exists an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ we have :

$$d(T_i x, Tx) < \frac{\epsilon'}{2} \text{ for all } x \in X.$$

So for $i > i_0$, if $d(T^{i-1} a, T^{i-1} x) \geqslant \frac{\epsilon'}{k_1 - k} = \epsilon$ we have :

$$d(T^i a, T^i x) = d(T_i T^{i-1} a, T_i T^{i-1} x) \leqslant d(T_i T^{i-1} a, TT^{i-1} a)$$
$$+ d(TT^{i-1} a, TT^{i-1} x) + d(TT^{i-1} x, T_i T^{i-1} x)$$
$$< \frac{\epsilon'}{2} + kd(T^{i-1} a, T^{i-1} x) + \frac{\epsilon'}{2} \leqslant k_1 d(T^{i-1} a, T^{i-1} x)$$

and if $d(T^{i-1} a, T^{i-1} x) < \frac{\epsilon'}{k_1 - k}$ it follows

$$d(T^i a, T^i x) < \frac{\epsilon}{2} + kd(T^{i-1} a, T^{i-1} x) + \frac{\epsilon'}{2} \leqslant \epsilon' + \frac{k\epsilon'}{k_1 - k} < \epsilon.$$

Thus in any case there exists an $m_0 \in \mathbb{N}$ such that for $n \geqslant m_0$

$$d(T^n x, T^n a) = d(T^n_{i_0} T^{i_0 - 1} x, T^n_{i_0} T^{i_0 - 1} \alpha)$$
$$< \max\{\epsilon, k_1^{n - i_0 + 1} d(T^{i_0 - 1} x, T^{i_0 - 1} a)\} = \epsilon.$$

Step II—We will now prove that for any $\epsilon > 0$ there exists an n_0 such that if $n \geqslant n_0$

$$d(T^n a, a) < \epsilon. \qquad \ldots(2.2)$$

Choose $\epsilon > 0$ and $\epsilon' = (k_1 - k)\epsilon$. Then there exists an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ it follows : $d(T_i x, Tx) < \epsilon'$ for all $x \in X$. So if $d(T^{i-1} a, a) \geqslant \frac{\epsilon'}{k_1 - k} = \epsilon$ we have

$$d(T^i a, a) = d(T^i a, Ta) = d(T_i T^{i-1} a, Ta) \leqslant d(T_i T^{i-1} a, TT^{i-1} a)$$
$$+ d(TT^{i-1} a, Ta) < \epsilon' + kd(T^{i-1} a, a) \leqslant k_1 d(T^{i-1} a, a)$$

and if

$$d(T^{i-1} a, a) < \frac{\epsilon'}{k_1 - k}$$

then

$$d(T^i a, a) < \epsilon' + kd(T^{i-1} a, a) < \frac{k_1 \epsilon'}{k_1 - k} < \epsilon.$$

So finally, in any case, there exists an $m_0 \in \mathbb{N}$ such that for $n > m_0$ it follows :

$$d(T^n a, a) < \max\{\epsilon, k_1^{n - i_0 + 1} d(T^{i_0 - 1} a, a)\} = \epsilon.$$

Step III—Now by (2.1) and (2.2) if for $\epsilon > 0$ we choose $N = \max\{n_0, m_0\}$, then for $n \geqslant N$ we have :

$$d(T^n x, a.) \leqslant d(T^n x, T^n a) + d(T^n a, a) < \epsilon + \epsilon$$

and the theorem is proved.

The following example and Example 2 of the next section show that we can not omit from the assumptions of Theorem 1 either that T is a contraction mapping or that the sequence $\{T_n\}$ converges uniformly to T.

Example 1—$X = \mathbb{R}$, $T_n x = 2x - 3 - 1/n$, $Tx = 2x - 3$, and the distance function d is the ordinary euclidean distance on the line. Thus $\{T_n\}$ converges uniformly to T but $\lim\limits_{n\to\infty} T^n x = +\infty$ if $x \geqslant 4$.

3. Convergence of the Sequence in a Locally Compact Metric Space

Theorem 2—Let (X, d) be a locally compact metric space, let $\{T_n\}$ be a sequence of contraction mappings of X into itself and let $T : X \to X$ be a contraction mapping with fixed point a. If the sequence $\{T_n\}$ converges pointwise to T then there exists an $r > 0$ and a positive integer m_0 such that $\lim\limits_{n\to\infty} T^n_m \; x = a$ for every $x \in K(a, r) \equiv \{x : d(a, x) \leqslant r\}$, whenever $m \geqslant m_0$.

Proof : Choose $r > 0$ such that the set $K(a, r)$ is a compact subset of X. From the equicontinuity property it follows that the sequence $\{T_n\}$ is uniformly convergent on $K(a, r)$. We choose m_0 such that if $n \geqslant m_0$ then

$$d(T_n x, Tx) < (1 - k) r$$

for all $x \in K(a, r)$, where $k < 1$ is the Lipschitz constant for T. Then if $n \geqslant m_0$ and $x \in K(a, r)$ we have :

$$d(T_n x, a) \leqslant d(T_n x, Tx) + d(Tx, a) \leqslant (1 - k) r + kr = r$$

and thus if $n \geqslant m_0$, T_n maps $K(a, r)$ into itself. Now we take $k_1 \in \mathbb{R}$ with $0 < k < k_1 < 1$, and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$, $N > m_0$ such that $n \geqslant N$ implies

$$d(T_n x, Tx) < \epsilon' = \epsilon(k_1 - k)$$

for all $x \in K(a, r)$, so that for these x and for $i - 1 \geqslant N > m \geqslant m_0$ if $d(T^{i-1}_m x, a) \geqslant \dfrac{\epsilon'}{k_1 - k}$ we have :

$$d(T^i_m \; x, a) \leqslant d(T_i \; T^{i-1}_m \; x, TT^{i-1}_m \; x) + d\;(TT^{i-1}_m \; x, Ta)$$

$$< \epsilon' + k\, d(T^{i-1}_m \; x, a) \leqslant k_1\, d(T^{i-1}_m \; x, a)$$

and if $d\,(T_m^{i-1}\,x, a) < \dfrac{\epsilon'}{k_1 - k}$ it follows

$$d\,(T_m^i\,x, a) < \epsilon' + k\,\frac{\epsilon'}{k_1 - k} < \epsilon.$$

Thus in both cases there exists an $n_0 \in \mathcal{N}$, $n_0 > N$, such that $n \geqslant n_0$ gives

$$d\,(T_m^n\,x, a) < \max\{\epsilon,\; k_1^{n-N}\, d\,(T_m^N\,x, a)\} = \epsilon.$$

We now give an example which shows that in non-locally compact spaces a sequence of contraction mappings $\{T_n\}$ may converge pointwise to a contraction mapping T with fixed point a, but for every $r > 0$ and every positive integer m_0, we can find an $x \in K(a, r)$ such that the sequence $\{T_{m_0}^n\,x\}$ does not converge to the point a. We proceed as in Nadler[1].

Example 2—Let X be an infinite dimensional separable or reflexive Banach space. Let X^* be the first conjugate of X and let $T = \{f \in X^* : \|f\| \leqslant 1\}$. Then T is weak* sequentially compact. Since X is infinite dimensional, there is a sequence $\{g_k\}$ of linear functionals in T which has no norm convergent subsequence. Let $\{g_{k_i}\}$ be a weak* convergent subsequence of $\{g_k\}$ and let g be the weak* limit of $\{g_{k_i}\}$. For each $i = 1, 2, 3, \ldots$ let

$$f_i = \frac{g_{k_i} - g}{\|g_{k_i} - g\|}.$$

The sequence $\{f_i\}$ is weak* convergent to the zero linear functional and $\|f_i\| = 1$ for all $i = 1, 2, 3, \ldots$

For each $i = 2, 3, \ldots$ let $a_i \in X$ such that $\|a_i\| = 1$ and

$$|f_i\,(a_i)| > \left(1 - \frac{1}{i^2}\right) \Big/ \left(1 - \frac{1}{i^3}\right)$$

and define $T_i : X \to X$ by

$$T_i\,x = \left(1 - \frac{1}{i^3}\right) f_i\,(x)\, a_{i+1}$$

for all $x \in X$. It is easily seen that $\{T_i\}$ converges point-wise to the zero mapping. Since

$$\|T_i\,x - T_i\,y\| = \left(1 - \frac{1}{i^3}\right) |f_i\,(x) - f_i\,(y)|\, \|a_{i+1}\| < \left(1 - \frac{1}{i^3}\right)$$

$$\|x - y\|$$

for all x and y in X, T_i is a contraction mapping for each $i = 1, 2, 3, \ldots$ and we have

$$T_m^n \; x = \left(1 - \frac{1}{n^3}\right)\left(1 - \frac{1}{(n-1)^3}\right) `\ldots' \left(1 - \frac{1}{m^3}\right) f_m\,(x)\, f_{m+1}$$

$$\times\; (a_{m+1})\; `\ldots'\; f_n\,(a_n)\; a_{n+1}$$

with

$$\|T_m^n \; x\| > \prod_{i=m+1}^{n} \left(1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{m^3}\right) | f_m(x) | \geqslant \tfrac{1}{2}\left(1 - \frac{1}{m^3}\right)$$

$$\times\; | f_m\,(x) | \,.$$

We conclude with an analogue of Theorem 3 of Nadler[1] which characterizes finite dimensional spaces.

Theorem 3—A separable or reflexive Banach space X is finite dimensional if and only if whenever a sequence of contraction mappings of X into X converges pointwise to a contraction mapping T with fixed point a, then there exists an $m_0 \in N$ and an $r > 0$ such that $\lim_{n \to \infty} T_m^n \; x = a$ for all $x \in K(a, r)$ whenever $m \geqslant m_0$.

PROOF : The condition is obviously sufficient, since every finite dimensional Banach space is locally compact and thus Theorem 2 applies. The converse assertion follows from Example 2.

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NOETHERIAN REGULAR RINGS

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In this paper, we obtain various characterizations of Noetherian regular rings.

1. INTRODUCTION

It is well known that in a commutative ring with identity, every radical ideal can be represented as an intersection of prime ideals. In general, the representation is not unique. Here we observe that this is true in the case of Noetherian regular rings (Throughout in this paper "regular" means "Von Neumann regular"). We characterize Noetherian regular rings by means of saturated sets. Using this characterization we also prove that a commutative ring with identity is Noetherian regular if and only if it is semiprime in which every non unit is a zero-divisor and the zero ideal is a product of a finite number of principal ideals generated by semiprimary elements.

Throughout R denotes a commutative ring with identity. We now recall some definitions.

A nonempty subset S in R is said to be saturated if for any $x, y \in R$, $x, y \in S$ if and only if $xy \in S$. A proper saturated set S is said to be maximal if S is not contained in any proper saturated set of R.

It is easy to see that maximal saturated sets always exist in R.

Let $S(R)$ denote the set of all saturated sets of R. For any collection of S_α's in $S(R)$, define $VS_\alpha = \{x \in R \mid xy = f_{\alpha_1} f_{\alpha_2} \dots f_{\alpha_n}$ for some $y \in R$ and $f_{\alpha_i} \in S_{\alpha_i}\}$. Then $(S(R), V, \cap)$ is a lattice with 'V' as the supremum and set intersection '$\cap$' as the infimum. For any $a \in R$, the principal saturated set generated by 'a' is the intersection of all saturated sets containing 'a' and is denoted by $[a)$. Infact $[a) = \{x \in R \mid xy = a^n$ for some $y \in R$ and $n \in Z^+ \cup \{0\}\}$. It can be easily seen that $[a)$ is the smallest saturated set containing a.

An ideal A of R is said to be a radical ideal if $A = \sqrt{A} = \{x \in R \mid x^n \in A$ for some $n \in Z^+\}$. All ideals are assumed to be proper. For any $a \in R$, the principal ideal generated by a is denoted by (a); $P(R)$ denotes the set of all prime ideals of R.

2. Noetherian Regular Rings

We obtain various characterizations of Noetherian regular rings. We shall begin with the following lemma.

Lemma 1—Let R be semiprime and let F be a maximal saturated set in R. If $\sum_{i=1}^{n} a_i \in F$ then $a_i \in F$ for some i.

Proof : Suppose $a_i \notin F$ for all $i = 1, 2, \ldots, n$. As F is a maximal saturated set, we get $a_i f_i = 0$ for some $f_i \in F$, $1 \leqslant i \leqslant n$. Therefore $(\prod_{1=i}^{n} f_i)\ a_i = 0$ for $i = 1, 2, \ldots, n$ and so $(\prod_{i=1}^{n} f_i)(\sum_{i=1}^{n} a_i) = 0$. Thus $0 \in F$, a contradiction. Hence $a_i \in F$ for some i.

Remark 1 : The above lemma shows that the complement of any maximal saturated set in a semiprime ring is always a prime ideal.

Theorem 1—R is regular if and only if R is semiprime satisfying any one of the following two conditions :

(i) the complement of every maximal saturated set in R is a maximal ideal;
(ii) the complement of every maximal ideal in R is a maximal saturated set.

Proof : It can be easily seen that the conditions (i) and (ii) are equivalent in a semiprime ring R. If R is regular then R is semiprime and every prime ideal in R is a maximal ideal and hence by the above remark R satisfies condition (i). Now we assume that R is a semiprime ring satisfying condition (i). Let $x \in R$ and $T(x) = \{y \in R \mid xy = 0\}$, $D(x) = \{y \in R \mid (x) + (y) = R\}$. If $T(x) \cap D(x) \neq \phi$ then we are through. Suppose $T(x) \cap D(x) = \phi$. It is easy to verify that $T(x)$ is an ideal and that $D(x)$ is a saturated subset of R. Let $\mathscr{S} = \{S \in S(R) \mid D(x) \subsetneq S, S \cap T(x) = \phi\}$. Partially order $\mathscr{S}$ by set-inclusion. By Zorn's lemma, $\mathscr{S}$ has a maximal element, say F. If $x \notin F$ then $(FV[x]) \cap T(x) \neq \phi$. Hence there is an element $y \in T(x)$ such that $y \in FV[x)$. As $xy = 0$ and $y \in FV[x]$ we have $fx^n = 0$ for some $f \in F$ so that $F \cap T(x) \neq \phi$ which is impossible. Therefore $x \in F$. Now we show that F is a maximal saturated set. Let $F \subset G$ for some saturated set G of R. Choose $z \in G - F$. Then $(FV[z]) \cap T(x) \neq \phi$ and so there exists an element $y \in T(x)$ such that $y \in FV[z)$. Since R is semiprime it now follows that $fzx = 0$ for some $f \in F$. Clearly $fx \in F \subset G$ and hence $0 = fzx \in G$. Consequently $G = R$. This shows that F is a maximal saturated set. Now by condition (i), $R-F$ is a maximal ideal and $x \notin R-F$, so $(R-F) + (x) = R$: i.e., $(y) + (x) = R$ for some $y \in R-F$, so $y \in D(x)$ which is impossible since $D(x) \subsetneq F$. Hence $T(x) \cap D(x) \neq \phi$. Thus every principal ideal is a direct summand and hence R is regular.

Theorem 2—R is Noetherian regular if and only if R is semiprime and (*) every radical ideal has a unique representation as an intersection of prime ideals.

PROOF : Suppose R is Noetherian regular. Then R is semiprime and R contains only a finite number of prime ideals. Let I be an ideal in R. Then $I = \bigcap_{i=1}^{n} P_i = \prod_{i=1}^{n} P_i$, where P_i's are prime ideals in R. If $I = \bigcap_{i=1}^{m} Q_i = \prod_{i=1}^{m} Q_i$ where Q_j's are prime ideals in R, then it can be easily seen that each $P_i = Q_j$ for some j and each $Q_j = P_i$ for some i. This shows that the representation is unique.

Conversely assume that R is semiprime satisfying condition (*). First we show that every radical ideal is principal. Let I be a radical ideal. Put $J = \cap \{P \in P(R) \mid I \nsubseteq P\}$. Since R is semiprime, clearly $IJ = \cap \{P \mid P \in P(R)\} = (0)$ so that $I \cap J = (0)$. We show that $I + J = R$. If $I + J \neq R$ then $I + J \subseteq P_0$ for some prime ideal P_0 of R. Then $J = \cap \{P \mid I \nsubseteq P\} = (\cap \{P \mid I \nsubseteq P\}) \cap P_0$ so that J has two representations. It now follows that $I + J = R$. As $I + J = R$ and $I \cap J = (0)$, clearly $I = (e)$ for some idempotent $e \in R$.

Now let I be any ideal. Then by above observation, we get $\sqrt{I} = (e)$ for some idempotent $e \in R$, and hence $I = (e)$. Thus every ideal is principal and also for each $a \in R$, $(a) = (e)$ for some idempotent $e \in R$. Hence R is Noetherian regular.

Remark 2 : In a principal ideal ring R, every radical ideal need not have a unique representation as an intersection of prime ideals. For example, in Z the zero ideal is a prime ideal which can also be expressed as the intersection of all non-zero prime ideals of Z. This shows that regularity is essential in the above theorem. We also note that semiprime rings need not satisfy the condition (*).

Remark 3 : Any ring R satisfying (*) need not be semiprime. For, in the ring of integers modulo 4, the ideal $(\bar{2})$ is the only radical prime ideal and $\bar{2}$ is a nonzero nilpotent element.

Definition—A nonzero element $a \in R$ is said to be an atom if for each $x \in R$, $x^n a^m = 0$ for some positive integers n, m or $a^n = xy$ for some positive integer n and for some $y \in R$.

Lemma 2—A nonzero element $a \in R$ is an atom if and only if $[a)$ is a maximal saturated set in R.

PROOF : Suppose 'a' is an atom. Let $[a) \subseteq F$ for some proper saturated set F of R. Let $x \in F$. Then $x^n a^m \neq 0$ for all $n, m \in Z^+$ so that $a^n = xy$ for some $n \in Z^+$ and $y \in R$. This shows that $x \in [a)$ and hence $F = [a)$; i.e , $[a)$ is a maximal saturated set.

Conversely assume that $[a)$ is a maximal saturated set. Let $x \in R$. If $x \in [a)$ then $a^n = xy$ for some $y \in R$ and so we are through. If $x \notin [a)$ then $[a) \vee [x) = [0)$ so that $0 = fg$ for some $f \in [a)$ and $g \in [x)$. Now $fy_1 = a^m$ and $gy_2 = x^n$ for some $y_1, y_2 \in R$ so that $x^n a^m = 0$ for some $n, m \in Z^+$. This shows that 'a' is an atom.

Lemma 3—If every prime ideal of R is a principal ideal generated by some idempotent element then every ideal is a principal ideal generated by some idempotent element.

Proof follows by applying Zorn's lemma.

Theorem 3—The following statements are equivalent :

(i) R is a Noetherian regular ring;

(ii) R is semiprime and every maximal saturated set is principally generated by some idempotent element;

(iii) R is semisimple and every maximal saturated set is principal.

PROOF : (i) $\Rightarrow$ (ii). Suppose (i) holds. Clearly R is semiprime. Let F be a maximal saturated set. Then by Theorem 1, $R-F$ is a maximal ideal and so $R-F=(e)$ for some idempotent $e \in R$. We observe that $1-e \in F$. We now show that $1-e$ is an atom. Let $x \in R$. Suppose $x(1-e) \neq 0$. Then $x \notin (e)$ and so $(x)+(e) = R$ since (e) is a maximal ideal. Now $1 = xy + ey$, for some $y, y_1 \in R$ so that $1-e = x(1-e)y$. This shows that $1-e$ is an atom. It now follows from Lemma 2 that $F = [1-e)$ where $1-e$ is an idempotent.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $a \in \cap \{M \mid M$ is a maximal ideal of $R\}$. Assume $a \neq 0$. Then $[a) \subsetneq [e)$ for some maximal saturated set $[e)$ where e is an idempotent. Clearly $1-e$ is not a unit and hence there exists a maximal ideal M such that $1-e \in M$. Now $aa_1 = e$ for some $a_1 \in R$. Since $a \in M$ it follows that $e \in M$ and hence we get $1 \in M$, a contradiction. Therefore we must have $a = 0$ which shows that R is semisimple and hence (iii) follows.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. We first show that R is regular. Lex $x \in R$. Write $T(x) = \{y \in R \mid xy = 0\}$ and $D(x) = \{y \in R \mid (x)+(y) = R\}$. If $T(x) \cap D(x) \neq \phi$ then we are done. Assume $T(x) \cap D(x) = \phi$. By the same argument as before, we can get a maximal saturated set F such that $D(x) \subsetneq F$ and $x \in F$. As F is a maximal saturated set, $F = [a)$ for some $a \in R$. Since R is semisimple, $P + (a) = R$ for some proper ideal P of R. Also $P + (a^n) = R$ for any $n \in Z^+$. In fact we can have $(p) + (a^n) = R$ for some $p \in P$. Clearly $p \notin F$. Since F is a maximal saturated set, R being semiprime we get ge $ap = 0$. Also $x \in F$ so that $(a^n) \subsetneq (x)$ and hence $(x) + (p) = R$; i.e., $p \in D(x) \subsetneq F$. This shows that $0 = ap \in F$, a contradiction. Therefore $T(x) \cap D(x) \neq \phi$; i.e., R is regular.

Finally we show that every prime ideal is principal. Let P be a prime ideal. Then $R-P$ is a maximal saturated set so that $R-P = [a)$ for some $a \in R$. Since R is regular $(a) = (e)$ for some idempotent $e \in R$. We observe that $P = (1-e)$ and therefore by Lemma 3 every ideal is a principal ideal generated by some idempotent element. Hence R is a Noetherian regular ring.

Remark 4 : In (ii) of the above theorem the condition that "every maximal saturated set is principally generated by an idempotent element" is essential. Also in (iii), the semisimplicity of the ring is essential. These facts are illustrated in the following example

Example—Consider $Z_{(p)}$, the localization of the ring of integers at some prime ideal (p). Clearly the local ring $Z_{(p)}$ is semiprime but not semisimple. It can be easily verified that p as an element in $Z_{(p)}$ is an atom in $Z_{(p)}$. Obviously p is not an idempotent in $Z_{(p)}$. In view of Lemma 2, $[p)$ is a maximal saturated set in $Z_{(p)}$. Since $Z_{(p)}$ is an integral domain, it follows that this is the only maximal saturated set in $Z_{(p)}$.

Definition—An element $x \in R$ is said to be semiprimary if $\sqrt{(x)}$ is a proper prime ideal in R.

Lemma 4—Let R be a semiprime ring in which every non-unit is a zero divisor. Let $x \in R$ be a semiprimary element. If there exists a non-zero element $y \in R$ such that $xy = 0$ then $(x) + (y) = R$. Moreover such *a* y is an atom in R.

PROOF : Suppose $(x) + (y) \neq R$. Then $x + y$ is a non-unit and so $(x+y)\, d = 0$ for some $d \neq 0$. Now $yd \in (x) \subseteq \sqrt{(x)}$ and so either $y \in \sqrt{(x)}$ or $d \in \sqrt{(x)}$. If $y \in \sqrt{(x)}$ then $y^n = xx_1$ for some $n \in Z^+$ and $x_1 \in R$. Clearly $y^{n+1} = 0$. Since R is semiprime we get $y = 0$, a contradiction. Therefore $y \notin \sqrt{(x)}$ and so $d \in \sqrt{(x)}$. Now $d^m = xx_2$ for some $m \in Z^+$ and $x_2 \in R$ and so $d^m y = yxx_2 = 0$. It follows that $dy = 0$ so that $xd = 0$. Thus we have $0 = xd^m = xxx_2 = x^2 x_2$ from which it follows that $xx_2 = 0$; i.e., $d = 0$, again a contradiction. Therefore $(x) + (y) = R$.

Now we prove that y is an atom. Let $z \in R$ and $zy \neq 0$. As $xzy = 0$ and $zy \neq 0$, by above argument we must have $(x) + (zy) = 0$. Therefore $1 = xx_1 + zyy_1$ so $y = yxx_1 + zy^2 y_1$ which shows that y is an atom.

Theorem 4—R is a Noetherian regular ring if and only if R is semiprime, every non-unit is a zero-divisor and the zero ideal is a product of a finite number of principal ideals generated by semiprimary elements.

PROOF : One implication is evident.

Conversely assume the stated conditions. Then $0 = a_1 a_2 \ldots a_n$ where a_i's $\in R$ and a_i's are semiprimary elements. Since each a_i is a non-unit, there exist $b_i \in R$ such that $a_i b_i = 0$ with $b_i \neq 0$ for $i = 1, 2, \ldots, n$. By the above lemma we have for each i, $(a_i) + (b_i) = R$ and each b_i is an atom. Clearly $(a_i) \cap (b_i) = 0$ and so we get $(a_i) = (e_i)$ and $(b_i) = (f_i)$ for some idempotents e_i, f_i in R. It is easy to see that each e_i is a semiprimary element. Also $e_i f_i = 0$ $(f_i \neq 0)$ so that f_i is an atom. Now $\sum_{i=1}^{n} (f_i) + (a_i) = R$ for $i = 1, 2, \ldots, n$. Therefore $\sum_{i=1}^{n} (f_i) + (a_1 \ldots a_n) = \sum_{i=1}^{n} (f_i) + (0)$

$= R$; i.e., $\sum_{i=1}^{n} (f_i) = R$. Without loss of generality we may assume that $f_i \neq f_j$ for $i \neq j$. Since f_i's are distinct idempotent atoms we get $f_i f_j = 0$ for $i \neq j$ so that $\sum_{i=1}^{n} (f_i) = (\sum_{i=1}^{n} f_i) = R$. Consequently $\sum_{i=1}^{n} f_i = 1$. Now let F be a maximal saturated set. Then $\sum_{i=1}^{n} f_i = 1 \in F$ and so by Lemma 1, $f_i \in F$ for some i. Again since f_i is an atom, by Lemma 2, $[f_i)$ is a maximal saturated set and therefore $F = [f_i)$ is a principal saturated set. Thus every maximal saturated set is a principal saturated set generated by an idempotent element. Hence by Theorem 3, R is a Noetherian regular ring. Hence proof of the theorem.

Remark 5 : In the above theorem, the condition that "R is semiprime" is essential. This can be seen by considering Z_4, the ring of integers modulo 4. In Z_4, 0 is a semiprimary element and $\bar{2}$ is the only non-unit which is a zero-divisor. But Z_4 is not regular.

Remark 6 : By considering the ring of integers, we can see that the condition "every non-unit is a zero divisor" cannot be dropped.

Remark 7 : It is well known that $\mathscr{P}(X)$, where X is an infinite set, is a Boolean ring which is not Noetherian. It is semiprime and every non-unit in $\mathscr{P}(X)$ is a zero-divisor. However, the zero ideal cannot be written as a finite product of principal ideals generated by semiprimary elements.

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DIFFERENTIAL SUBORDINATION AND CONFORMAL MAPPINGS I

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It is well-known that if $N(z)$ and $D(z)$ are analytic in the unit disc, $N(0) = D(0) = 0$ and D maps the open unit disc onto a multi-sheeted domain that is starlike with respect to the origin then Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > 0$. In this paper we give a sufficient condition on $D(z)$ which can guarantee that Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > \alpha > 0$. A few interesting applications of this result are also given.

1. INTRODUCTION

Libera[4] showed that if $N(z)$ and $D(z)$ are analytic in the unit disc, $N(0) = D(0) = 0$ and D maps the open unit disc onto a multi-sheeted starlike region with respect to the origin then Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > 0$. Various generalisations of this result can be found in the literature. However the examples $N(z) = zf'(z)$ where $f(z)$ is univalently convex and $D(z) = f(z)$ (which is necessarily univalently starlike of order 2^{-1}) suggest that there may exist some conditions on N and D so that Re $(N'(z)/D'(z)) > 0$ in the unit disc implies Re $(N(z)/D(z)) > \alpha > 0$ for $|z| < 1$. But just the stronger assumption that D is convex will not be sufficient for the required implication. This will be clear if we consider the following example. Let $D(z) = z/(1+z)$ (and hence D is convex) and $N(z)$ be determined by $(N'(z)/D'(z)) = (1+z)/(1-z)$. Then $(N(z)/D(z)) = ((1+z)/2z) \log ((1+z)/(1-z))$ and this function has a limit 0 as z goes to -1 through reals. Hence there cannot be a positive constant α such that Re $(N(z)/D(z)) > \alpha$ for $|z| < 1$. Thus it is interesting to ask whether there exists a condition on $D(z)$ for our required implication. The aim of this paper is to find at least one such condition and to give some interesting applications for this result.

2. PRELIMINARIES

Definition 1—A function $f(z)$ regular in the open unit disc U is said to be α convex for some real α if

$$(f(z)f'(z)/z) \neq 0 \text{ in } U \text{ and}$$

$$\text{Re}\,((1-\alpha)\, zf'(z)/f(z) + \alpha\,(1+(zf''(z)/f'(z)))) > 0 \text{ in } U \text{ (See Mocanu[6]).}$$

We note that α-convex functions are always starlike and for $\alpha \geqslant 1$ these functions are in fact convex. We use the notation $K(\alpha)$ to denote the class of all α-convex functions in U.

Definition 2—A function $f(z)$ regular in U is said to be α-close-to-convex $(\alpha \geqslant 0)$ if

$$(f(z)\, f'(z)/z) \neq 0 \text{ in } U \text{ and}$$

there exists a starlike function $\varphi(z)$ in U such that

$$\text{Re}\,((1-\alpha)\,(zf'(z)/\varphi(z)) + \alpha\,((zf'(z))'/\varphi'(z))) > 0.$$

The class of all such functions will be denoted by $C(\alpha)$ and this class generalises the class $K(\alpha)$ for $\alpha \geqslant 0$.

Definition 3—A regular function $f(z)$ in U is said to be a Bazilevic function of type α (α real) if

$$(f(z)/z) \neq 0 \text{ in } U \text{ and}$$

$$\text{Re}\,\{f'(z)/(f(z))^{1-\alpha}\} > 0 \text{ in } U.$$

We choose a suitable branch for the power function used here. This class of functions will be denoted by $B(\alpha)$ (Bazilevic[1]).

Definition 4—Let $r = r_1 + ir_2$, $s = s_1 + is_2$. Let $b = e^{i\beta}$ with $|\beta| < \pi/2$. We shall say that a map $\psi : \mathbb{C}^2 \to \mathbb{C}$ belongs to $S(D)$ if

(1) $\psi(r, s)$ is continuous

(2) $(b, 0) \in D$ and $Re\,\psi(b, 0) > 0$.

(3) Re $\psi(r_2 i, s_1) \leqslant 0$ when $(r_2 i, s_1) \in D$ and

$$s_1 \leqslant -\tfrac{1}{2}(1 - 2r_2 \sin\beta + r_2^2)/\cos\beta$$

See Lewandowski[3].

Theorem A—Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the open unit disc U of the complex plane. Suppose for each $z_0 \in U$ there exists a function $\psi_{z_0} \in S(\mathbb{C}^2)$ such that Re $\psi_{z_0}(p(z_0), z_0\, p'(z_0)) > 0$ then Re $p(z) > 0$ for $z \in U$.

PROOF : Essentially the proof is the same as that of Lemma 2.1 of Lewandowski[3]. But since we have slightly changed the hypothesis we shall complete the proof.

With the same notations as in the proof of Lemma 2.1 of Lewandowski[3] we assume that $p(z) = (1 + w(z))\,(1 - w(z))^{-1} \cos\beta + i \sin\beta$ and claim $|w(z)| < 1$ or Re $p(z) > 0$. If there exists a point $z_0 = e^{i\theta_0} \in U$ such that $\max\limits_{|z| \leqslant r_0} |w(z)| = |w(z_0)| = 1$, we get as in Lewandowski[3] that

$$p(z_0) = \theta i,\; z_0\, p'(z_0) = d$$

where θ and d are real and $d \leqslant -\frac{1}{2}(1 - 2\theta \sin \beta + \theta^2)/\cos \beta$. But by hypothesis for this z_0 there exists a $\psi_{z_0} \in U(\mathbb{C}^2)$ with the property that Re $\psi_{z_0}(p(z_0), z_0 p'(z_0)) > 0$ and Re $\psi_{z_0}(\theta i, d) \leqslant 0$ with θ and d as above. Because of this contradiction $|w(z)| < 1$ and so Re $p(z) > 0$ in U establishing our claim.

3. Main Results

Theorem B—Let α be real and non-negative, $M(z)$, $N(z)$ regular functions in U with $M(0) = N(0) = 0$ and $(M'(0)/N'(0)) = 1$. Let $N(z)$ satisfy

$$\text{Re}\,(N(z)/zN'(z)) > \delta \ (0 \leqslant \delta < 1).$$

If

$$\text{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] > 0 \ (z \in U)$$

then

$$\text{Re}\,(M(z)/N(z)) > \delta\alpha/(2 + \delta\alpha) \ (z \in U).$$

Proof: Let $\beta = \delta\alpha/(2 + \delta\alpha)$ and consider

$$p(z) = (1 - \beta)^{-1}[(M(z)/N(z)) - \beta].$$

This $p(z)$ is regular in U and $p(0) = 1$. We set

$$\lambda(z) = N(z)/zN'(z)$$

and observe that by hypothesis Re $(\lambda(z)) > \delta$.

A simple computation shows that

$$(1 - \alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = \beta + (1-\beta)\,p(z) + \alpha(1-\beta)\,\lambda(z)\,zp'(z)$$

$$= \psi_z(p(z), zp'(z))$$

where $\psi_z(r, s) = (1 - \beta)r + \beta + \alpha(1 - \beta)\lambda(z)s$. Now ψ_z is continuous in $\mathbb{C}^2$, Re $\psi_z(1, 0) = 1 > 0$ and for all $(ir_2, s_1) \in \mathbb{C}^2$ with $s_1 \leqslant -(1 + r_2^2)/2$ we have

$$\text{Re}\,\psi_z(ir_2, s_1) = \beta + \alpha(1 - \beta)s_1 \text{Re}\,\lambda(z)$$

$$\leqslant \beta - \alpha(1 - \beta)\frac{1 + r_2^2}{2}\text{Re}\,\lambda(z)$$

$$\leqslant \beta - \alpha\frac{(1 - \beta)\delta}{2} = 0.$$

Hence for each z, $\psi_z \in S(\mathbb{C}^2)$ and

$$\text{Re}\,\psi_z(p(z), zp'(z)) > 0.$$

Hence by Theorem A, Re $p(z)) > 0$ and hence

$$\operatorname{Re}(M(z)/N(z)) > \beta.$$

This proves our theorem.

Corollary 1—If M and N are regular in U, $M(0) = N(0) = 0$, $M'(0)/N'(0)=1$ and $N(z)$ satisfies Re $(N(z)/zN'(z)) > \delta$ then

$$\operatorname{Re}(M'/N') > 0 \text{ implies } \operatorname{Re}(M/N) > \delta/(2+\delta).$$

PROOF : Take $\alpha = 1$ in Theorem B.

Corollary 2—If $f(z) \in B(m)$ where m is a positive integer then Re $(f(z)/z)^m > 1/(1+2m)$.

PROOF : Choose $M(z) = (f(z))^m$ and $N(z) = z^m$. Then $(M'(z)/N'(z)) = (f(z))^{m-1} f'(z)/z^{m-1}$ and $(M'(0)/N'(0)) = 1 > 0$. By Corollary 1 since $f \in B(m)$, Re $(M/N) > \delta/(2+\delta)$ whenever $(1/m) > \delta$. But δ can bc chosen as near $1/m$ as we please and so we can allow $\delta \to 1/m$ from below. Thus $(\delta/(2+\delta) \to 1/(1+2m)$ and we have established our claim.

We can draw a very interesting conclusion from the following theorem.

Theorem C—If ν and c are positive integers, $f(z)$ is a normalized analytic function in U with the property that

$$\operatorname{Re}\{f'(z)/(f(z)/z)^{1-\gamma}\} > -1/2(\gamma + c)$$

then the function $F(z)$ defined by

$$F^{\gamma}(z) = \frac{\gamma + c}{z^c}\int_0^z t^{c-1}(f(t))^{\gamma}\,dt$$

belongs to $B(\gamma)$.

PROOF : We shall write

$$p(z) = F'(z)/(F(z)/z)^{1-\gamma}$$

and after some simplification find

$$g = \frac{f'}{\left(\dfrac{f}{z}\right)^{1-\gamma}} = p + \frac{1}{\gamma + c}\,zp'.$$

Now we shall put $k = 1/2(\gamma + c)$ and consider

$$(g+k)/(1+k) = p/(1+k) + zp'/(\gamma + c)(1+k) + k/1+k$$
$$= \psi(p, zp')$$

where

$$\psi(r, s) = r/(1+k) + s/(\gamma+c)(1+k) + k/(1+k).$$

Clearly $\psi \in S(\mathbb{C}^2)$ because for $s_1 \leqslant \dfrac{-(1+r_2^2)}{2}$ we have

$$\operatorname{Re}\psi(ir_2, s_1) = s_1/(\gamma+c)(1+k) + k/(1+k)$$

$$\leqslant \frac{-(1+r_2^2)}{2(1+k)(\gamma+c)} + \frac{k}{1+k} \leqslant 0.$$

$$[\gamma + c = 1/2k \text{ and } -(1+r_2^2) \leqslant -1].$$

Hence we can take $\psi_z = \psi$ for all $z \in U$ and find that $\operatorname{Re}\psi_z(p, zp') > 0$. Hence $\operatorname{Re}(p(z)) > 0$ and so $F(z) \in B(\gamma)$ as required.

Remark : Under the hypothesis f need not belong to $B(\gamma)$ and so Theorem C is an improvement of a theorem in Singh[7]. Taking $\gamma = c = 1$ we see that if a normalised analytic function f satisfies $\operatorname{Re}(f'(z)) > -\frac{1}{4}$ then the corresponding F satisfies $\operatorname{Re} F'(z) > 0$ and hence is univalently close-to-convex. This result is of special interest since it extends an earlier result due to Libera[4]. Incidentally this result provides an example of a non-univalent function for which the Libera transform is univalent. Using similar arguments we can improve several other interesting results available in the literature. We will illustrate this as follows. The proofs will be omitted.

Theorem D—If $f(z)$ normalised analytic function in U satisfying

$$\operatorname{Re}[(1-\alpha)(f(z)/z) + \alpha f'(z)] > 0 \ (\alpha \geqslant 1)$$

then

$$\operatorname{Re}(f(z)/z) > \alpha/(\alpha+2) \text{ and } \operatorname{Re}(f'(z)) > (\alpha-1)/(\alpha+2).$$

The above theorem improves some results found in Chichra[2].

Theorem E—If $f \in C(\alpha)$ and the corresponding φ satisfies

$$\operatorname{Re}(\varphi(z)/z\varphi'(z)) > \delta$$

then $\operatorname{Re}(zf'(z)/\varphi(z)) > \delta\alpha/(2+\delta\alpha)$.

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ON A QUATERNION SUBMANIFOLDS OF CO-DIMENSION-2

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In this paper we have studied some properties of a Quaternion submanifolds of co-dimension-2. We have shown that a (1, 1)-tensor field defined by $f=B^{-1} F B$ possess the (f, g, u, v, λ) structure. We have also shown that tensor field defined by $f''' = \frac{F^* + G^* + H^*}{\sqrt{3}}$ admits an almost complex structure.

1. PRELIMINARIES

Integrability conditions of an almost quaternion structure have been obtained by Yano and Kon[1]. Upadhyay[2] has studied some theorems on metric-3 structure. Hamoui[3] has shown that a submanifold of co-dimension-2 admits (F, U, V, u, v, λ) $(G, U', V', u', v', \lambda)$, $(H, U'', V'', u'', v'', \lambda)$ and a most general structure. In this paper we have shown that a (1, 1), tensor field defined by $f = B^{-1} FB$ also possess the (f, g, u, v, λ) structure. Some other important results have also been obtained.

A Quaternion manifold M^{4n} admlts a set of 3-tensor F^*, G^*, H^* of type (1,1), satisfying[1]:

$$F^{*2} = -I,\ G^{*2} = -I,\ H^{*2} = -I \qquad \ldots(1.1)$$

(a) $F^* = G^* H^* = -H^* G^*$

(b) $G^* = F^* H^* = -H^* F^*$.

(c) $H^* = F^* G^* = -G^* F^*$. ...(1.2)

Let g^* be the Hermitian matric then, we have[3]:

$$g^* (F^* X^*, F^* Y^*) = g^* (X^*, Y^*) \qquad \ldots(1.3)$$

where X^*, Y^* are arbitrary vector fields in M^{4n}.

2. STRUCTURES IN M^{4n-2}

Let B represent the immersion say $i : M^{4n-2} \to M^{4n}$.

If C and D are mutually orthogonal unit normals. Then transformation $F^* BX$ of BX by F^* is expressed by[3] :

$$F^* BX = BFX + u(X) C + v(X) D. \quad \ldots (2.1)$$

Here X is an arbitrary vector field of M^{4n-2} & u. v, are 1 forms. F is a (1, 1) tensor field of M^{4n-2}. Since $F^*\ C$ & $F^*\ D$ are orthogonal to C & D respectively, hence

$$\left.\begin{array}{ll} \text{(a)} & g^*(F^* C, C) = -g^*(C, F^* C) = 0 \\ \text{(b)} & g^*(F^* D, D) = -g^*(D, F^* D) = 0. \end{array}\right\} \quad \ldots (2.2)$$

Now corresponding to structures F^*, G^*, H^*, the vector fields U, U', U'', V, V', V'' and form $u, u', u'', v, v'\ v''$ and a function λ such that[3]

$$\left.\begin{array}{ll} \text{(a)} & F^* C = -BU + \lambda D \\ \text{(b)} & F^* D = -BV - \lambda C. \end{array}\right\} \quad \ldots (2.3)$$

$$\begin{array}{ll} \text{(a)} & G^* BX = BGX + u'(X) C + v'(X) D \\ \text{(b)} & G^* C = -BU' + \lambda D \\ \text{(c)} & G^* D = -BV' - \lambda C. \end{array} \quad \ldots (2.4)$$

$$\begin{array}{ll} \text{(a)} & H^* BX = BHX + u''(X) C + v''(X) D \\ \text{(b)} & H^* C = -Bu'' + \lambda D. \\ \text{(c)} & H^* D = -Bv'' - \lambda C. \end{array} \quad \ldots (2.5)$$

If E be the Induced Riemannian connection in M^{4n-2} then we have Gauss and Weingarten equations as follows :

$$D^*_{BX} BY = BE_x Y + M(X, Y) C + L(X, Y) D \quad \ldots (2.6)$$

where C & D are symmetric bilinear functions in M^{4n-2}.

$$D^*_{BX}\ D = -B\,'L(X) - {}'K(X) C \quad \ldots (2.7)$$

where KX is third fundamental tensor

$$\text{(a)} \quad g('M(X), Y) = M(X, Y); \quad \ldots (2.8)$$

$$\text{(b)} \quad g('L(X), Y) = L(X, Y) \quad \ldots (2.9)$$

since B is the Jacobian map of i

such that $B ; T_p(M^{4n-2}) \to T_{ip}(M^{4n})$.

Suppose a tensor field N on M^{4n} which does not belong to $T(M^{4n})$ so N is now every where tangent to M^{4n-2}. Since B is one-one, B^{-1} also exists and a form N^* in M^{4n} (Sinha and Sharma[8]).

$$BB^{-1} = I,\ B^{-1} B = -I + N^* \otimes N,$$

$$N^* B = 0,\ B^{-1} N = 0,\ N^* N = 1. \qquad \ldots(2.10)$$

3. Submanifolds of Almost Quaternion Structure

Theorem 3.1—Let M^{4n-2} be a submanifold in a quaternion manifold M^{4n}. M^{4n}. Then a tensor field f admits (f, U, u, v, λ) —Structure.

Proof : Let us put[8]

$$f \text{ def } B^{-1} F^* B,$$

then

$$f^2 (X) = B^{-1} F^* BB^{-1} F^* BX: \qquad \ldots(3.1)$$

which in view of (2.1), (2.20) and (1.1) yields

$$f^2 (X) = -X + u(X)U + v(X)V \qquad \ldots(3.2)$$

also using (2.1), (9.3), (2.10) and (1.1), we can easily show that

$$u(fX) = \lambda v(X),\ v(fX) = -\lambda u(X)$$

$$f(U) = -\lambda V, f(V) = -\lambda U,\ v(U) = 0$$

$$u(U) = 1 - \lambda^2,\ v(V) = 1 - \lambda^2,\ u(V) = 0.$$

Theorem 3.2—The submanifolds M^{4n-2} of a quaternion manifold M^{4n} admits the following structure

$$(f', U', V', u', v', \lambda); (f'', U'', V'', u'', v'', \lambda)$$

Proof : Let us define $f' = B^{-1} G^* B, f'' = B^{-1} H^* B$ respectively and using (1. 1), (2.1), (2.3), (2.4), (2.5) and (2.10), this can be easily shown.

Theorem 3.3—The tensor field f''' defined by

$$''' = \frac{F^* + G^* + H^*}{\sqrt{3}}$$, admits an almost complex structure.

Proof : We can easily show by using (1,2) that $f'''^2 = -I$

$$\begin{aligned} g^* (f''' X, f''' Y) = 1/3\, [& g^* (F^* X, F^* Y) + g^* (G^* X, G^* Y) \\ & + g^* (H^* X, H^* Y) + g^* (F^* X, G^* Y) \\ & + g^* (F^* X, H^* Y) + g^* (G X, F^* Y) \\ & + g^* (G^* X, H^* Y) + g^* (H^* X, F^* Y) \\ & + g^* (H^* X, G^* Y)]. \end{aligned}$$

Making use of (1.1) – (1.3) & (2.1); we get

$$g^* (f''' X, f''' Y) = g^* (X, Y) \text{ and } g^* (f X, f Y) = g^* (F X, F Y).$$

4. General Proerties

Theorem 4.1—In M^{n-2} submanifold of a quaternion manifold

M^{4n}, If $(E_X\ F)(Y) = u(X)\ 'MY - v(Y)'\ L(X)$, then

$(D^*_{BX}\ F^*)(BY) - (D^*_{BY})(BX) = 0$, and similar results for G^* and H^*

Proof :—Using (2.1), (2,2) – (2.8), (2.10) and making use of $(E_X\ F)(Y) = v(Y)\ 'L(X) - u(X)\ 'M(Y)$, we find

$$(D^*_{BX}\ F^*)(BY) - (D^*_{BY}\ F^*)(BX) = 0. \qquad \text{...(4.1)}$$

Similarly it can be shown for G^* & H^* structures.

Theorem 4.2—In a submanifold M^{4n-2} of a quaternion manifold M^{4n}, we have

$$N^*\ D^*_{BX}\ F^*\ C = 0,\ N^*\ D^*_{BX}\ G^*\ C = 0,\ N^*\ D^*_{BX}\ H^*\ C = 0;$$

and

$$N^*\ D^*_{BX}\ F^*\ D = 0,\ N^*\ D^*_{BX}\ G^*\ D = 0,\ N^*\ D^*_{BX}\ H^*\ C = 0.$$

Proof : Using (2.3) — (2.2), (2.7) & (2.8) we get

(a) $B^{-1}(D^*_{BX}\ F^*\ C) = -E_X\ U - \lambda' L(X)$

(b) $B^{-1}\ (D^*_{BX}\ F^*\ D = -E_X\ V - \lambda' M(X)$

(c) $B^{-1}\ (D^*_{BX}\ G^*\ C) = -E_X\ U' - \lambda' L(x)$

(d) $B^{-1}\ (D^*_{BX} G^*\ D) = -E_x\ V' - \lambda' M(X)$

(e) $B^{-1}\ (D^*_{BX}\ H^*\ C) = -E_X\ U'' - \lambda' L(X)$

(f) $B^{-1}(D^*_{BX}\ H^*\ D) = -E_X\ V'' - \lambda' L(X)$ ...(4.2)

using these equation and keeping in view of (2.10) the theorem follows.

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ON ALMOST CONTINUOUS FUNCTIONS

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Some characterizations of almost continuous functions in the sense of Singal[16] are obtained. It is shown that every nearly almost open and almost weakly continuous function is almost continuous in the sense of Husain[6].

1. INTRODUCTION

Singal and Singal[16] introduced and investigated the notion of almost continuous functions. Husain[6] introduced the notion of almost continuous functions. Long and Carnahan[10] pointed out that these two notions of almost continuous functions are independent of each other. The purpose of the present paper is to obtain some characterizations of almost continuity in the sense of Singal and to show that every nearly almost open and almost weakly continuous function is almost continuous in the sense of Husain.

2. PRELIMINARIES

Throughout the present paper, X and Y always mean topological spaces and by $f: X \rightarrow Y$ we denote a single valued function. Let A be a subset of X. The closure of A and the interior of A are denoted by Cl (A) and Int (A), respectively. A subset A is said to be α-open[12] (resp. preopen[11]) if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ (resp. $A \subset \text{Int}(\text{Cl}(A))$). A subset A is said to be semi-open[9] (resp. semi-preopen[3]) if there exists an open (resp. preopen) set U such that $U \subset A \subset \text{Cl}(U)$. It is shown that a subset A is semi-open (resp. semi-preopen) if and only if $A \subset \text{Cl}(\text{Int}(A))$ (resp. $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$). The family of all α-open (resp. semi-open, preopen and semi-preopen) sets of X is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$ and $SPO(X)$).

Lemma 2.1—For a topological space X, the following properties hold :

(i) $\alpha(X) = PO(X) \cap SO(X)$ and (ii) $PO(X) \cup SO(X) \subset SPO(X)$.

PROOF : This follows easily from the definitions.

The complement of an α-open (resp. semi-open, preopen) set is said to be α-closed (resp. semi-closed, preclosed). The intersection of all α-closed (resp. semi-closed, preclosed) sets containing A is called the α-closure[2] (resp. semi-closure[4], preclosure[5])

and is denoted by α Cl (A) (resp. s Cl (A), P cl (A)). A subset A is said to be regular open if $A =$ Int (Cl (A)). The complement of a regular open set is said to be regular closed.

3. Almost Continuity in the Sense Singal

Definition 3.1—A function $f: X \to Y$ is said to be almost continuous[16] (briefly a.c.S.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset$ Int (Cl (V)).

Singal and Singal[16] showed that a function $f: X \to Y$ is a.c.S. if and only if $f^{-1}(V)$ is open (resp. closed) in X for every regular open (resp. regular closed) set V of Y. This characterization is very useful and will be utilized in the sequel.

Theorem 3.2—The following are equivalent for a function $f: X \to Y$:

(a) f is a.c.S.

(b) Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)) for every $V \in SPO(Y)$.

(c) Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)) for every $V \in SO(Y)$.

(d) $f^{-1}(V) \subset$ Int $(f^{-1}$ (Int (Cl (V)))) for every $V \in PO(Y)$.

Proof: (a) $\Rightarrow$ (b): Let $V \in SPO(Y)$. By Theorem 2.4 of Andrijević[3], Cl (V) is regular closed in Y. Since f is a.c.S., f^{-1} (Cl (V)) is closed in X and we obtain Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)).

(b) $\Rightarrow$ (c): Since $SO(Y) \subset SPO(Y)$, this is obvious.

(c) $\Rightarrow$ (a): Let F be any regular closed set of Y. Then $F =$ Cl (Int (F)) and hence $F \in SO(Y)$. Therefore, we have Cl $(f^{-1}(F)) \subset f^{-1}$ (Cl (F)) $= f^{-1}(F)$. Hence $f^{-1}(F)$ is closed and f is a.c.S.

(a) $\Rightarrow$ (d): Let $V \in PO(Y)$. Then $V \subset$ Int (Cl (V)) and Int (Cl (V)) is regular open. Since f is a.c.S., f^{-1} (Int (Cl (V))) is open in X and hence $f^{-1}(V) \subset f^{-1}$ (Int (Cl (V))) $=$ Int $(f^{-1}$ (Int (Cl (V)))).

(d) $\Rightarrow$ (a): Let V be any regular open set of Y. Then $V \in PO(Y)$ and hence $f^{-1}(V) \subset$ Int $(f^{-1}$ (Int (Cl (V)))) $=$ Int $(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is open in X and hence f is a.c.S.

Lemma 3.3—For a subset V of Y, the following properties hold:

(a) αCl $(V) =$ Cl (V) for every $V \in SPO(Y)$.

(b) P cl $(V) =$ Cl (V) for every $V \in SO(Y)$.

(d) s cl $(V) =$ Int (Cl (V)) for every $V \in PO(Y)$.

Proof: (a) Let $V \in SPO(Y)$. Then $V \subset$ Cl (Int (Cl (V))) and by Theoren 2.2 of Andrijević[2] we have α Cl $(V) = V \cup$ Cl (Int (Cl (V))) $=$ Cl (V).

(b) This follows from Theorem 2.4 of El-Deeb *et al.*[5].

(c) Let $V \in PO(Y)$. Then $V \subset \text{Int}(\text{Cl}(V))$ and by Theorem 1.5 of Andrijević[3], we have $s\,\text{Cl}(V) = V \cup \text{Int}(\text{Cl}(V)) = \text{Int}(\text{Cl}(V))$.

Corollary 3.4—The following are equivalent for a function $f : X \to Y$:

(a) f is a.c.S.

(b) $\text{Cl}(f^{-1}(V)) \subset f^{-1}(\alpha\,\text{Cl}(V))$ for every $V \in SPO(Y)$.

(c) $\text{Cl}(f^{-1}(V)) \subset f^{-1}(P\,\text{cl}(V))$ for every $V \in SO(Y)$.

(d) $f^{-1}(V) \subset \text{Int}(f^{-1}(s\,\text{Cl}(V)))$ for every $V \in PO(Y)$.

PROOF : This is an immediate consequence of Theorem 3.2 and Lemma 3.3.

Long and Carnahan[10] showed that if $f : X \to Y$ is open a.c.S. then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every open set V of Y. Recently, Allam *et al.*[1] have improved this result as follows if : $f : X \to Y$ is open a.c.S. then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in PO(Y)$. The following corollary is the further improvement of the previous result.

Corollary 3.5—If a function $f : X \to Y$ is open and a.c.S., then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SPO(Y)$.

PROOF : Since f is open, $\text{Cl}(f^{-1}(S)) \supset f^{-1}(\text{Cl}(S))$ for every subset S of Y. Therefore, this follows immediately from Theorem 3.2.

Definition 3.6—A function $f : X \to Y$ is said to be almost open[15] if $f(U) \subset \text{Int}(\text{Cl}(f(U)))$ for every open set U of X.

Mashhour *et al.*[11] called an almost open function preopen. Theorem 11 of Rose[15] states that $f : X \to Y$ is almost open if and only if $f^{-1}(\text{Cl}(V)) \subset \text{Cl}(f^{-1}(V))$ for every open set V of Y. It is shown in Theorem 14 of Rose[15] that $f : X \to Y$ is almost open and a.c.S. if and only if $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every open set V of Y.

Theorem 3.7—A function $f : X \to Y$ is almost open and a.c.S. if and only if $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SO(Y)$.

PROOF : *Necessity*—Let $V \in SO(Y)$. Since f is a.c.S., by Theorem 3.2 $\text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$. Since f is almost open, we have

$$f^{-1}(\text{Cl}(V)) = f^{-1}(\text{Cl}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(V)).$$

Therefore, we obtain $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SO(Y)$.

Sufficiency—It follows from Theorem 11 of Rose[15] that f is almost open. Let F be any regular closed set of Y. Then $F = \text{Cl}(\text{Int}(F))$ and hence $F \in SO(Y)$. By

the hypothesis, $\mathrm{Cl}(f^{-1}(F)) = f^{-1}(\mathrm{Cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is closed in X. Therefore, f is a.c.S.

Corollary 3.8—A function $f: X \to Y$ is almost open and a.c.S if and only if $\mathrm{Cl}(f^{-1}(V)) = f^{-1}(\mathrm{Cl}(V))$ for every $V \in \alpha(Y)$.

PROOF: Since $\alpha(Y) \subset SO(Y)$ and every open set is α-open, this follows from Theorem 3.7 and Theorem 14 of Rose[15].

The following question will be raised naturally: can $SO(Y)$ in Theorem 3.7 be replaced by $PO(Y)$? Actually, it is shown in Corollary 2.4 of Allam *et al.*[1] that if $f: X \to Y$ is OPEN a.c.S. then $\mathrm{Cl}(f^{-1}(V)) = f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$. However, the answer is negative under the condition that f is almost open a.c.S. as the following example shows.

Example 3.9—Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b, c\}, \{a, c, d\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a\}, \{c\}\}$. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \phi, \{x, y\}, \{z\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined as follows: $f(a) = x$, $f(b) = y$ and $f(c) = f(d) = z$. Then f is continuous (hence a.c.S.) almost open but not open. There exists $V = \{y, z\} \in PO(Y)$ such that $\mathrm{Cl}(f^{-1}(V)) \neq f^{-1}(\mathrm{Cl}(V))$.

4. Almost Continuity in the Sense of Husain

Definition 4.1—A function $f: X \to Y$ is said to be almost continuous[6] (briefly a.c.H.) if for each $x \in X$ and each open set V of Y containing $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighbourhood of x.

It is easily proven that $f: X \to Y$ is a.c.H. if and only if $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$, that is, $f^{-1}(V) \in PO(X)$ for every open set V of Y. Mashhour *et al.*[11] called a.c.H. functions *precontinuous*.

Definition 4.2—A function $f: X \to Y$ is said to be almost weakly continuous[7] if $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ for every open set V of Y.

Levine[8] defined $f: X \to Y$ to be weakly continuous if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Cl}(V)$ and showed that $f: X \to Y$ is weakly continuous if and only if $f^{-1}(V) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y. Therefore, almost weak continuity is implied by both weak continuity and almost continuity in the sense of Husain.

Theorem 4.3—The following are equivalent for a function $f: X \to Y$:

(a) f is almost weakly continuous.

(b) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$.

(c) $P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$.

(d) $P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y.

PROOF : (a) $\Rightarrow$ (b) : Let $V \in PO(Y)$. By utilizing Theorem 3.1 of Noiri[13], we obtain $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))) = f^{-1}(\mathrm{Cl}(V))$.

(b) $\Rightarrow$ (c) : Let $V \in PO(Y)$. By utilizing Theorem 1.5 of Andrijević[3], we obtain $P\mathrm{cl}(f^{-1}(V)) = f^{-1}(V) \cup \mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$.

(c) $\Rightarrow$ (d) : This is obvious since every open set is preopen.

(d) $\Rightarrow$ (a) : Let V be any open set of Y. We have $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ and hence by Theorem 3.1 of Noiri[13] f is almost weakly continuous.

Definition 4.4—A function $f: X \rightarrow Y$ is said to be nearly almost open[14] if there exists an open basis $\mathscr{B}$ for the topology on Y such that $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every $V \in \mathscr{B}$.

Every almost open function is nearly almost open but not conversely by Example 3 of Rose[14]. Rose[15] showed that every almost open weakly continuous function is a.c.H. Moreover, Rose[14] showed that nearly almost open weakly continuous functions are a.c.H. On the other hand, Noiri[13] showed that every almost open almost weakly continuous function is a.c.H. The following theorem is an improvement of the previous results.

Theorem 4.5—If a function $f: X \rightarrow Y$ is nearly almost open and almost weakly continuous, then f is a.c.H.

PROOF : Since f is nearly almost open, there exists an open basis $\mathscr{B}$ for the topology on Y such that $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every $V \in \mathscr{B}$. Let W be any open set of Y. There exists a subfamily $\mathscr{B}_0$ of $\mathscr{F}$ such that $W = \cup\{V \mid V \in \mathscr{B}_0\}$. Therefore, we obtain

$$f^{-1}(W) = \bigcup_{V\in\mathscr{B}_0} f^{-1}(V) \subset \bigcup_{V\in\mathscr{B}_0} \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))) \subset \bigcup_{V\in\mathscr{B}_0}$$

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) \subset \mathrm{Int}(\mathrm{Cl}(\bigcup_{V\in\mathscr{B}_0} f^{-1}(V)))$$

$$= \mathrm{Int}(\mathrm{Cl}(f^{-1}(W))).$$

This shows that $f^{-1}(W) \in PO(X)$ and hence f is a.c.H.

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ON THE MULTIVALENT FUNCTIONS

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The object of the present paper is to derive some sufficient conditions for p-valently close-to-convexity, p-valently starlikeness and p-valently convexity.

1. INTRODUCTION

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N)$$

which are regular in $D = \{z \mid |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p-valently convex if and only if

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > 0 \text{ in } D.$$

We denote by $C(p)$ the subclass of $A(p)$ consisting of all p-valently convex functions in D.

A function $f(z)$ in $A(p)$ is said to be p-valently starlike if and only if

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \text{ in } D.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of all functions which are p-valently starlike in D.

A function $f(z)$ in $A(p)$ is said to be p-valently close-to-convex, if there exists a p-valently starlike function $g(z) \in A(p)$ for which $f(z)$ satisfies the following

$$\operatorname{Re} \frac{z f'(z)}{g(z)} > 0 \text{ in } D.$$

We denote by $K(p)$ the subclass of $A(p)$ consisting of all functions which are p-valently close-to-convex in D.

Every p-valently starlike function is p-valently close-to convex, and every p-valently close-to-convex function is p-valent in D[2,10,4]. Ozaki[7] (Theorem 3) proved that if

$f(z) \in A(p)$ and

$$1 + \mathrm{Re}\, \frac{z f''(z)}{f'(z)} < \frac{k + p + 1}{2} \quad \text{in } D$$

then $f(z)$ is at most k-valent in D.

Moreover, by using Umezawa's[11] result (Theorem 6), we have that if $f(z) \in A(p)$ and

$$1 + \mathrm{Re}\, \frac{z f''(z)}{f'(z)} < p + \tfrac{1}{2} \text{ in } D$$

then $f(z)$ is convex of order at most p in one direction in D, and at most p-valent in D.

2. Preliminaries

Lemma 1—Let $f(z) \in A(p)$ and if there exists a $(p - k + 1)$-valently starlike function $g(z) = \sum_{n=n-p+1}^{\infty} b_n z^n$, $(b_{p-k+1} \neq 0)$ that satisfies

$$\mathrm{Re}\, \frac{z f^{(k)}(z)}{g(z)} > 0 \text{ in } D$$

then $f(z)$ is p-valently close-to-convex in D.

Proof : From Theorem 8 of Nunokawa[4], we have

$$\mathrm{Re}\, \frac{z f'(z)}{G(z)} > 0 \text{ in } D$$

where $G(z)$ is p-valently starlike in D.

This shows that $f(z) \in K(p)$ and $f(z)$ is p-valently close-to-convex in D.

Lemma 2—Let $f(z) \in A(p)$ and suppose that there exists a positive integer k for which

$$\left| \arg \frac{f^{(k)}(z)}{z^{p-k}} \right| < \frac{\pi}{2} \alpha \text{ in } D \qquad \text{...(1)}$$

where $1 \leq k \leqq p$ and $0 < \alpha \leq 1$.

Then we have

$$\left| \arg \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} \right| < \pi \alpha \text{ in } D.$$

Proof : We easily have

$$\frac{f^{(k-1)}(z)}{z f^{(k)}(z)} = \int_0^1 \frac{f^{(k)}(tz)}{f^{(k)}(z)}\, dt$$

(*equation continued on p.* 579)

$$= \frac{z^{p-k}}{f^{(k)}(z)} \int_0^1 t^{p-k} \frac{f^{(k)}(tz)}{(tz)^{p-k}} \, dt \qquad \ldots(2)$$

and it easily follows that

$$\left| \arg t^{p-k} \frac{f^{(k)} \, tz}{(tz)^{p-k}} \right| = \left| \arg \frac{f^{(k)} \, tz}{(tz)^{p-k}} \right| < \frac{\pi}{2} \alpha \text{ in } D$$

for $0 \leqq t \leqq 1$.

Then the integral

$$\int_0^1 t^{p-k} \frac{f^{(k)}(tz)}{(t\,z)^{p-k}} \, dt$$

lies in the same convex sector $\{w \| \arg w \mid < \frac{\pi}{2} \alpha\}$.

Therefore, from (1) and (2), we have

$$\left| \arg \frac{f^{(k-1)}(z)}{zf^{(k)}(z)} \right| \leqq \left| \arg \frac{z^{p-k}}{f^{(k)}(z)} \right| + \left| \arg \int_0^1 t^{p-k} \frac{f^{(k)} \, tz}{(tz)^{p-k}} \, dt \right.$$

$$< \frac{\pi}{2} \alpha + \frac{\pi}{2} \alpha = \pi \alpha \text{ in } D.$$

This completes our proof.

Lemma 3—Let $f(z) \in A(p)$ and suppose there exists a positive integer k for which

$$k + \operatorname{Re} \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0 \text{ in } D$$

where $1 \leqq k \leqq p$.

Then we have

$$k - 1 + \operatorname{Re} \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D.$$

We owe this lemma to Nunokawa[4] (Lemma 9).

3. Statement of Results

Theorem 1—Let $f(z) \in A(p)$ and suppose that the exists a positive integer k for which

$$k + \mathrm{Re}\ \frac{z f^{(k-1)}(z)}{f^{(k)}(z)} < \beta \text{ in } D$$

where $1 \leqq k \leqq p$ and $p < \beta \leqq p + \frac{1}{2}$.

Then we have $f(z) \in C(p)$ or $f(z)$ is p-valently close-to-convex in D.

Proof : Let us put

$$H(z) = \frac{1}{\beta - p}\left\{\beta - k - \frac{z f^{(k-1)}(z)}{f^{(k)}(z)}\right\} = \frac{z g'(z)}{g(z)}.$$

Then we have $H(0) = 1$ and $\mathrm{Re}\ H(z) > 0$ in D. This shows that $g(z)$ is univalently starlike in D.

Applying the same method as in Nunokawa and Owa[5,6], we easily have

$$\frac{1}{\beta - p} \log \frac{z^{p-k}}{f^{(k)}(z)}\ {}_pA_k = \log \frac{g(z)}{z}$$

where ${}_pA_k = p(p-1)(p-2)\ldots(p-k+1)$.

It follows that

$$\frac{f^{(k)}(z)}{{}_pA_k\, z^{p-k}} = \left(\frac{g(z)}{z}\right)^{p-\beta}.$$

Applying the result by Komatu[7] and Robinson[8], we have

$$\frac{f^{(k)}(z)}{{}_pA_k\, z^{p-k}} = \left(\frac{g(z)}{z}\right)^{p-\beta} \prec \left(\frac{1}{1-z}\right)^{2(p-\beta)} \text{ in } D$$

where the symbol $\prec$ denotes subordination.

Then we have

$$\left|\arg \frac{f^{(k)}(z)}{{}_pA_k\, z^{p-k}}\right| = \left|\arg \frac{f^{(k)}(z)}{z^{p-k}}\right| \leqq 2(\beta - p)\sin^{-1}|z|$$

$$< 2(\beta - p)\frac{\pi}{2} \leqq \frac{\pi}{2} \text{ in } D.$$

This shows that

$$\mathrm{Re}\ \frac{f^{(k)}(z)}{z^{p-k}} = \mathrm{Re}\ \frac{z f^{(k)}(z)}{z^{p-k+1}} > 0 \text{ in } D. \qquad \ldots(3)$$

On the other hand, $g(z) = z^{p-k+1}$ is $(p-k+1)$-valently starlike in D and therefore we have

$$\mathrm{Re}\ \frac{z f^{(k)}(z)}{g(z)} > 0 \text{ in } D.$$

From Lemma 1, we have $f(z) \in K(p)$. This completes our proof.

Remark 1 : Applying the same method as in the proof of Theorem 6 of Nunokawa[3] and from (3), we can also prove Theorem 1.

Theorem 2—Let $f(z) \in A(p)$ and suppose that there exists a positive integer k for which

$$k + \operatorname{Re} \frac{z f^{(k+1)}(z)}{f^{(k)}(z)} < \beta \text{ in } D$$

where $2 \leqq k \leqq p$ and $p < \beta \leqq p + \frac{1}{4}$.

Then we have $f(z) \in C(p)$ and $f(z) \in S(p)$, or $f(z)$ is p-valently convex in D and $f(z)$ is p-valently starlike in D.

PROOF : Applying the same method as in the proof of Theorem 1, we have

$$\frac{f^{(k)}(z)}{pAkz^{p-k}} = \left(\frac{g(z)}{z} \right)^{p-\beta} \prec \left(\frac{1}{1-z} \right)^{2(p-\beta)} \text{ in } D. \qquad \ldots(4)$$

From the assumption of Theorem 2 and (4), we have

$$\left| \arg \frac{f^{(k)}(z)}{pAk\, z^{p-k}} \right| = \left| \arg \frac{f^{(k)}(z)}{z^{p-k}} \right|$$

$$< \frac{\pi}{2}\, 2(\beta - p) \leqq \frac{\pi}{4} \text{ in } D. \qquad \ldots(5)$$

From Lemma 2 and (5), we have

$$\left| \arg \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} \right| < \frac{\pi}{2} \text{ in } D.$$

This shows that

$$\operatorname{Re} \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D$$

and it follows that

$$k - 1 + \operatorname{Re} \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D.$$

Applying Lemma 3 over again, we have

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > 0 \text{ in } D$$

and

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \text{ in } D.$$

This comletes our proof.

Applying the same method as in the proof of Theorem 2, we have

Theorem 3—Let $f(z) \in A(p)$ and suppose

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} < p + \tfrac{1}{4} \text{ in } D$$

where $2 \leqq p$.

Then $f(z)$ is p-valently starlike in D.

Remark 2 : Singh and Singh[9] (Theorem 6) proved the following theorem :

Theorem — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in D and

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} < \tfrac{3}{2} \text{ in } D$$

then $f(z)$ is starlike in D.

The proof of this theorem is simple but it is not generalized to multivalent functions. Therefore, the author gives finally the following conjecture :

Conjecture—Let $(z) \in A(p)$ and suppose

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} < p + \tfrac{1}{2} \text{ in } D.$$

Then $f(z)$ is p-valently starlike in D.

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THE HANKEL-CLIFFORD TRANSFORMATION ON CERTAIN SPACES OF ULTRADISTRIBUTIONS

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In this paper we introduce certain spaces of testing functions ${}^pS^{\beta}_{\mu,\alpha}$ contained in H_μ. The elements of the dual spaces are ultradistributions. The Hankel-Clifford transform h_μ for $\mu \geqslant 0$ is a continuous linear operator in spaces of these type. The generalized Hankel-Clifford transform h'_μ is defined as a continuous linear mapping between the dual spaces. The developed theory is applied to find classes of existence of classical and generalized solutions for a Cauchy problem of the Bessel type operator $B_\mu = Dx^{\mu+1}\, Dx^{-\mu}$.

1. INTRODUCTION

Méndez[5] introduced the space of testing function H_μ that consists of all infinitely differentiable functions ψ on $I = (0, \infty)$ such that

$$\sup_{x\in I} |\, x^m\, D^n\, (x^{-\mu}\, \psi\, (x))\, | < \infty \qquad \text{for every } m, n \in N$$

and its dual H'_μ to extend the classical Hankel-Clifford transformation to distributions of slow growth. More recently Gonzalez[2] considers some new function spaces, which are similar to the space studied by Lee[4]. The Hankel-Clifford transform acts on these spaces as a continuous linear mapping. Also, Gonzalez[2] analyzes a Cauchy problem for the systems of equations containing the Bessel-type operator $B_\mu = Dx^{\mu+1}\, Dx^{-\mu}$. He obtains classes of uniqueness and existence of generalized solution for such problem. The definition of a new convolution compatible with the Hankel-Clifford transformation allows to him to obtain an explicit expression of the solutions.

In this paper, according to the ideas of Romieu[7], Gelfand and Shilov[1], Pathak and Pandey[6] and others, we introduce spaces of ultradifferentiable functions, denoted by ${}^pS^{\beta}_{\mu,\alpha}$, with a structure similar to the spaces defined by Sanchez[8]. The Hankel-Clifford transform is a continuous linear mapping on this spaces. Therefore, the generalized transformation is also a linear and continuous mapping on the corresponding dual spaces.

We prove some properties of the spaces ${}^{p}S^{\beta}_{\mu,\alpha}$ and study how the differentiable operators D, $P_\mu = x^{\mu+1} Dx^{-\mu}$ and B_μ act on them. Moreover, certain spaces of multipliers are defined.

Finally, we consider the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = P(B_\mu)\, u(x,t)$$

$$u(x,t_0) = f(x)$$

where $u(x,t)$ is an unknown vector function, $u(x,t) = \{u_j(x,t)\}$ and P a matrix of polynomials. New classes of existence of classical and generalized functions are obtained.

2. Some Spaces of Testing Functions and Their Duals

In this section we introduce certain spaces of testing functions, subspaces of H_μ, which have a structure similar to those defined by Sànchez[8].

2.1. *The Space* ${}^{p}S_{\mu,\alpha,A}$

Let $\alpha \geqslant 0$, $\mu \in R$ and $p \in N$. We define the function space ${}^{p}S_{\mu,\alpha,A}$ as the collection of all complex valued smooth functions ψ defined on I such that

$$|\, x^m D^q (x^{-\mu} \psi(x)) \,| \leqslant C_{q,\delta} (A+\delta)^m (pm)!^\alpha$$

for every $q \in N$ and $\delta > 0$. $C_{q,\delta}$ are constants depending on ψ.

${}^{p}S_{\mu,\alpha,A}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|_{q,\delta} = \sup_{\substack{x\in I\\ m\in N}} \frac{|\, x^m D^q (x^{-\mu}\psi(x)) \,|}{(A+\delta)^m (pm)!^\alpha}$$

for every $q \in N$ and $\delta > 0$, each $\|\ \|_{q,\delta}$ is a seminorm on ${}^{p}S_{\mu,\alpha,A}$, and the collection $\Gamma = \{\|\ \|_{q,\delta}\}_{q\in N,\ \delta>0}$ is a multinorm because each $\|\ \|_{0,\delta}$ is a norm. Since the systems of seminorms Γ and $\Gamma_1 = \{\|\ \|_{q,1/n}\}_{q,n\in N}$ are equivalent, the space ${}^{p}S_{\mu,\alpha,A}$ equiped with the topology generated by Γ_1, is a countable multinormed space.

We now list some interesting properties of the space ${}^{p}S_{\mu,\alpha,A}$.

Property 2.1.1—${}^{p}S_{\mu,\alpha,A} \subset H_\mu$, the inclusion being continuous.

Property 2.1.2—${}^{p}S_{\mu,\alpha,A}$ is complete and therefore a Fréchet space.

To prove the last assertion it is enough to use Property 2.1.1, since H_μ is a complete space.

As a consequence of the above results, ${}^{p}S_{\mu,\alpha,A}$ is clearly a space of testing functions. Its dual, $({}^{p}S_{\mu,\alpha,A})'$, is a space of generalized functions.

Property 2.1.3—If $\alpha > 0$, then $D(I) \subset {}^pS_{\mu,\alpha,A}$ and the topology of $D(I)$ is stronger that the topology induced by ${}^pS_{\mu,\alpha,A}$ in $D(I)$.

PROOF : If $\psi \in D(I)$ one has

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_q (A+\delta)^m (pm)!^\alpha \left(\frac{L}{A+\delta} \right)^m (pm)!^{-\alpha}$$

for $m, q \in N$ and $\alpha > 0$, where $L = \sup \{x : x \in \operatorname{supp} \psi\}$ and $C_q = \sup\limits_{0<x<L} |D^q x^{-\mu} \psi(x)|$. Hence

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_\delta C_q (A+\delta)^m (pm)!^\alpha$$

with $C_\delta > \left(\frac{L}{A+\delta} \right)^m (pm)!^{-\alpha}$, for $m \in N$. Consequently, $D(I) \subset {}^pS_{\mu,\alpha,A}$ (both algebraically and topologically).

From the above result, the nontriviality of ${}^pS_{\mu,\alpha,A}$ follows provided that $\alpha > 0$. This space is dense in $E(I)$.

On the other hand, if $\alpha = 0$ and $\sup\limits_{x \in I} | x^m D^q (x^{-\mu} \psi(x)) | < C_{q,\delta} (A+\delta)^m$, for $\delta > 0$ and $m, q \in N$, then $\psi \in {}^pS_{\mu,0,A}$. Hence, ${}^pS_{\mu,0,A}$ coincides with the space $H_{\mu,0,A}$ defined by Gonzalez[2]. He denoted this space by $B_{\mu,A}$ due to its relation with the space $\beta_{\mu,A}$ introduced by Zemanian[13].

Property 2.1.4—(a) ${}^pS_{\mu,\alpha,A} \subset H_{\mu,\alpha p, p^{p\alpha}A}$

(b) If $p > 1$, $H_{\mu,\alpha,A} \subset {}^pS_{\mu,\alpha,A}$

(c) $H_{\mu,\alpha,A} \subset {}^1S_{\mu,r\alpha,A}$ with $r > 1$.

All inclusions are continuous.

Recall that the space $H_{\mu,\alpha,A}$ (Gonzalez[2]) consists of all smooth functions $\psi(x)$ defined on $0 < x < \infty$ such that

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant (A+\delta)^m m^{m\alpha} C_{q,\delta}$$

for $\delta > 0$ and $m, q \in N$. Then the proof of the Property 2.1.4 follows simply by using Stirling's formula.

Every inclusion transforms bounded sets into bounded sets, therefore it is continuous.

Property 2.1.5—${}^pS_{\mu+k,\alpha,A}$ is contained in ${}^pS_{\mu,\alpha,A}$, for each $k \in N$, the inclusion being continuous.

PROOF : Assume $k = 1$ and choose $\psi \in {}^pS_{\mu+1,\alpha,A}$. One then has $\sup\limits_{x \in I}$ $| x^m D^q (x^{-\mu} \psi(x)) | \leqslant \sup\limits_{x \in I} | x^{m+1} D^q x^{-\mu-1} \psi(x) | + q \sup\limits_{x \in I} | x^m D^{q-1} x^{-\mu-1} \psi(x) |$

$\leqslant C_{q,\delta}\,(A + \delta)^{m+1}\,((m+1)\,p)!^{\alpha} + qC_{q-1,\delta}\,(A + \delta)^{m}\,(pm)!^{\alpha}$

for $\delta > 0$, $m \in N$ and $q \in N - \{0\}$. Moreover :

$$\sup_{x \in I} |\, x^m\, x^{-\mu}\, \psi(x)\,| = \sup_{x \in I} |\, x^{m+1}\, x^{-\mu-1}\, \psi(x)\,| < C_{0,\delta}\,(A + \delta)^{m+1}\,(p\,(m+1))!^{\alpha}.$$

Making use of Stirling's formula it can easily be seen that ψ is in ${}^{p}S_{\mu,\alpha,A}$.

The proof is completed by induction on k.

The following result permit one to define a countable union space.

Property 2.1.6—If $0 < A_1 < A_2$ then ${}^{p}S_{\mu,\alpha,A_1} \subset {}^{p}S_{\mu,\alpha,A_2}$, the inclusion being continuous.

Hence, the union space can be defined as

$${}^{p}S_{\mu,\alpha} = \bigcup_{A=1}^{\infty} {}^{p}S_{\mu,\alpha,A}$$

which is endowed with the inductive limit topology.

${}^{p}S_{\mu,\alpha}$ is a space of testing functions and its dual, $({}^{p}S_{\mu,\alpha})'$, is a space of generalized functions.

2.2. *The space* ${}^{p}S_{\mu}^{\beta,B}$.

Let μ be a real number, $\beta \geqslant 0$ and $B > 0$. We define ${}^{p}S_{\mu}^{\beta,B}$ as the space of complex valued smooth functions $\psi(x)$ on $I = (0, \infty)$ such that

$$\sup_{x \in I} |\, x^m\, D^q\,(x^{-\mu}\,\psi(x))\,| \leqslant C_{m,\rho}\,(B + \rho)^q\,(pq)!^{\beta}$$

for every $m, q \in N$ and $\rho > 0$. $C_{m,\rho}$ are constants depending on ψ.

It can be easily seen that the set $\Gamma = \{\|\ \|^{m,\rho}\}_{m \in N,\, \rho > 0}$ represents a system of norms on ${}^{p}S_{\mu}^{\beta,B}$. Here

$$\|\psi\|^{m,\rho} = \sup_{\substack{x \in I \\ q \in N}} \frac{|\, x^m\, D^q\,(x^{-\mu}\,\psi(x))\,|}{(B + \rho)^q\,(pq)!^{\beta}} \quad \text{for every } m \in N \text{ and } \rho > 0.$$

We now study several properties of the space ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.2.1—${}^{p}S_{\mu}^{\beta,B}$ is contained in H_μ and the inclusion is continuous.

Property 2.2.2—${}^{p}S_{\mu}^{\beta,B}$ is a Fréchet space.

${}^{p}S_{\mu}^{\beta,B}$ is a space of testing functions, and its dual $\left({}^{p}S_{\mu}^{\beta,B} \right)'$, is a complete space of generalized functions, equiped either with the weak topology or with the strong topology.

If $H_{\mu}^{\beta,B}$ denotes the space defined by González[2], consisting of all the smooth functions ψ defined on $I = (0, \infty)$ such that

$$\sup_{x \in I} | x^m D^q (x^{-\mu} \psi (x)) | \leqslant C_{m,\rho} (B + \rho)^m m^{m\beta}$$

for $m, q \in N$ and $\rho > 0$, one has :

Property 2.2 3—(a) ${}^{p}S_{\mu}^{\beta,B} \subset H_{\mu}^{p\beta, Bp^{p\beta}}$

(b) if $p > 1$, then $H_{\mu}^{\beta,B} \subset {}^{p}S_{\mu}^{\beta,B}$

(c) $H_{\mu}^{\beta,B} \subset {}^{1}S_{\mu}^{r\beta,B}$, where $r > 1$

the inclusions being continuous.

Our next result is an useful test of convergence in ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.2.4—Let $\{\psi_\nu\}_{\nu \in N}$ be a sequence. If a positive constant $C_{m,\rho}$ exists for any $m \in N$, and $\rho > 0$ such that $\|\psi_\nu\|^{m,\rho} < C_{m,\rho}$ for every $\nu \in N$ and $D^q (x^{-\mu} \psi_\nu (x)) \to 0$ as $\nu \to \infty$, uniformly on $x \in (0, \epsilon)$, for every $q \in N$ and $\epsilon > 0$, then, $\psi_\nu \to 0$ as $\nu \to \infty$ in ${}^{p}S_{\mu}^{\beta,B}$.

PROOF : Let $m \in N$ and $\rho, \eta > 0$. Choose ρ' such that $0 < \rho' < \rho$. In these conditions,

$$\|\psi_\nu\|^{m,\rho} < C_{m,\rho'} \neq 0, \quad \text{for every } \nu \in N$$

where $C_{m,\rho'}$ is a constant. Moreover, there exists $q_0 \in N$ such that

$$\left(\frac{B + \rho'}{B + \rho} \right)^q < \frac{\eta}{C_{m,\rho'}} \quad \text{for every } q \geqslant q_0. \quad \text{Hence}$$

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | < C_{m,\rho'} (B + \rho')^q (pq)!^\beta < \eta (B + \rho)^q (pq)!^\beta,$$

$$\text{for } q \geqslant q_0.$$

By taking $q < q_0$ and $x > C_{m+1,\rho}/\eta$, we have

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | = \frac{| x^{m+1} D^q (x^{-\mu} \psi_\nu (x)) |}{x}$$

(equation continued on p. 588)

$$\leqslant \frac{1}{x} \|\psi_\nu\|^{m+1,\rho} (B+\rho)^q (pq)!^\beta$$

$$< \eta (B+\rho)^q (pq)!^\beta$$

and in virtue of uniform convergence, there exists a $\nu_0 \in N$ such that

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | \leqslant \eta (B+\rho)^q (pq)!^\beta,$$

for $\nu \geqslant \nu_0$, $q < q_0$ and $x < C_{m+1,\rho}/\eta$.

Therefore

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | \leqslant \eta (B+\rho)^q (pq)!^\beta$$

for $\nu \geqslant \nu_0$, $x \in I$ and $q \in N$

or in other terms: $\|\psi_\nu\|^{m,\rho} \leqslant \eta$, for $\nu \geqslant \nu_0$. Hence, $\psi_\nu \to 0$ as $\nu \to \infty$ in ${}^pS_\mu^{\beta,B}$.

Property 2.2.5—${}^pS_{\mu+k}^{\beta,B}$ is contained in ${}^pS_\mu^{\beta,B}$, and the topology of ${}^pS_{\mu+k}^{\beta,B}$ is stronger that the one induced in it by ${}^pS_\mu^{\beta,B}$, for every $k \in N$.

The proof of this property is similar to that of Property 2.1.5.

Property 2.2.6—If $0 < B_1 < B_2$ then ${}^pS_\mu^{\beta,B_1} \subset {}^pS_\mu^{\beta,B_2}$, the inclusions being continuous.

This allows to define the countably union space

$${}^pS_\mu^\beta = \bigcup_{B=1}^{\infty} {}^pS_\mu^{\beta,B}.$$

${}^pS_\mu^\beta$ equiped with the inductive limit topology is a space of testing functions and its dual $\left({}^pS_\mu^\beta \right)'$, is a space of generalized functions.

2.3. The space ${}^pS_{\mu,\alpha,A}^{\beta,B}$

Let μ be a real number, $\alpha, \beta \geqslant 0$ and $A, B > 0$. The space ${}^pS_{\mu,\alpha,A}^{\beta,B}$ consists of all the smooth complex valued functions ψ defined on $0 < x < \infty$ such that

$$\sup_{x \in I} | x^m D^q (x^{-\mu} \psi (x)) | < C_{\delta,\rho} (A+\delta)^m (pm)!^\alpha (B+\rho)^q (pq)!^\beta$$

for every $m, q \in N$, $\delta, \rho > 0$ with $C_{\delta,\rho}$ is a constant depending on ψ.

We consider on the space ${}^pS_{\mu,\alpha,A}^{\beta,B}$ the norms

$$\|\psi\|_{\delta}^{\rho} = \sup_{\substack{x\in I\\ m\in N\\ q\in N}} \frac{| x^m D^q (x^{-\mu} \psi (x)) |}{(A + \delta)^m (pm)!^{\alpha} (B + \rho)^q (pq)!^{\beta}} \quad \text{for all } \delta, \rho > 0.$$

The systems of norms $\Gamma_1 = \left\{\| \ \|_{\delta}^{\rho}\right\}_{\delta,\rho>0}$ and $\Gamma_2 = \left\{\| \ \|_{1/n}^{1/n}\right\}_{n\in N}$ are equivalent. The space ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ endowed with the topology generated by Γ_2, is a countably multinormed space.

We now present some properties of this space that are similar to those of ${}^{p}S_{\mu,\alpha,A}$ and ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.3.1—${}^{p}S_{\mu,\alpha,A}^{\beta,B} \subset H_{\mu}$ and the inclusion being continuous.

Property 2.3.2—${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ is a Fréchet space.

Therefore ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ is a space of testing functions, and its dual $\left({}^{p}S_{\mu,\alpha,A}^{\beta,B}\right)'$ is a space of generalized functions that is complete with the weak and the strong topologies.

We denote by $H_{\mu,\alpha,A}^{\beta,B}$ the space defined by Gonzàlez[2] that is constituted by the smooth complex valued functions ψ on I satisfying

$$\sup_{x\in I} | x^m D^q (x^{-\mu} \psi (x)) | \leqslant C_{\delta,\rho} (A + \delta)^m m^{m\alpha} (B + \rho)^q q^{q\beta}$$

for $m, q \in N$ and $\delta, \rho > 0$.

Property 2.3.3—(a) ${}^{p}S_{\mu,\alpha,A}^{\beta,B} \subset H_{\mu,p\alpha,A p^{p\alpha}}^{p\beta,B p^{p\beta}}$

(b) if $p > 1$, thin $H_{\mu,\alpha,A}^{\beta,B} \subset {}^{r}S_{\mu,\alpha,A}^{\beta,B}$

(c) $H_{\mu,\alpha,A}^{\beta,B} \subset {}^{p}S_{\mu,r\alpha,A}^{r_1\beta,B}$, with $r > 1$ and $r_1 > 1$.

The inclusions are continuous.

Property 2.3.4—${}^{p}S_{\mu+k,\alpha,A}^{\beta,B}$ is contained in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ and the topology of ${}^{p}S_{\mu+k,\alpha,A}^{\beta,B}$ is stronger than the topology induced in it by ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$, for every $k \in N$.

The following test of convergence can be proved as in Property 2.2.4.

Property 2.3.5—Let $\{\psi_v\}_{v\in N}$ be a sequence in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$. If for each $\delta, \rho > 0$

(i) there exists a positive constant $C_{\delta,\rho}$ such that $\|\psi_\nu\|_\delta^\rho < C_{\delta,\rho}$, for every $\nu \in N$ and

(ii) $D^q(x^{-\mu}\phi_\nu(x))$ converges to 0, as $\nu \to \infty$, uniformly on $y \in (0, \epsilon)$ for every $q \in N$, $\epsilon > 0$

then $\psi_\nu \to 0$, as $\nu \to \infty$, in ${}^pS^{\beta,B}_{\mu,\alpha,A}$.

Property 2.3.6—If $0 < A_1 < A_2$ and $0 < B_1 < B_2$, then

$$ {}^pS^{\beta,B_1}_{\mu,\alpha,A_1} \subset {}^pS^{\beta,B_2}_{\mu,\alpha,A_2} $$

the inclusion being continuous.

We can construct the countable union space:

$$ {}^pS^{\beta}_{\mu,\alpha} = \bigcup_{\substack{A=1\\B=1}}^{\infty} {}^pS^{\beta,B}_{\mu,\alpha,A} $$

${}^pS^{\beta}_{\mu,\alpha}$ is equiped with the inductive limit topology.

2.4 The space ${}^p\hat{S}^{\beta,B}_{\mu}$:

Let μ be a real number, $\beta \geq 0$ and $B > 0$. We define the function space:

$$ {}^p\hat{S}^{\beta,B}_{\mu} = \left\{ \psi \in C\,{}^pS^{\beta,B}_{\mu} : C_{k,\rho} \leq C'_\rho,\ \text{for } k \in N,\ \rho > 0 \right\}. $$

This space is endowed with the topology induced in it by ${}^pS^{\beta,B}_{\mu}$.

2.5 On the Nontriviality of the Spaces of type ${}^pS^{\beta}_{\mu,\alpha}$ is related to nontriviality of spaces of type $H^{\beta}_{\mu,\alpha}$ studied by González[2].

This last author proved that the mapping

$$ S^{\beta}_{\alpha} \to H^{\beta}_{\mu,\alpha} $$

$$ \psi(y) \to y^{\mu}\psi(y) $$

is linear and continuous. The properties of spaces of type S^{β}_{α} (see Gelfand and Shilov[1]), 2.1.5, 2.2.3 and 2.3.3 show that the following spaces are nontrivial:

(a) ${}^{p}S_{\mu,\alpha,A}$ and ${}^{p}S^{\beta,B}$ for every $\alpha \geqslant 0$, $\beta \geqslant 0$, $B > 0$ and $A > 0$,

(b) ${}^{p}S^{\beta,B}_{\mu,\alpha,H}$ for $\alpha > 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta > 1$; $A, B > 0$.

(c) ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ for $A, B > 0$ and $\alpha + \beta \geqslant 1$.

(d) ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ for $p > 1$, $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and $A, B > \gamma$, where γ is a positive constant.

3. Operational Calculus

In this section we show that derivation, multiplication by x, and some important linear differential operators, can be defined and are continuous on the previously introduced spaces.

Property 3.1—The mapping $x^n : {}^{p}S^{\beta}_{\mu,\alpha} \to {}^{p}S^{\beta}_{\mu,\alpha}$ is linear and continuous for every $n \in N$.

Proof : In effect, assuming $n = 1$ and taking, for example, ψ in ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$, then

$$\begin{aligned}
| x^m D^q (x^{-\mu} x \psi(x)) | &\leqslant | x^{m+1} D^q (x^{-\mu} \psi(x)) | \\
&\quad + q \, | x^m D^{q-1} (x^{-\mu} \psi(x)) | \\
&\leqslant C_{\delta,\rho} \{ (A + \delta)^{m+1} (p(m+1))!^{\alpha} (B + \rho)^q (pq)!^{\beta} \\
&\quad + q (A + \delta)^m (pm)!^{\alpha} (B + \rho)^{q-1} (p(q-1))!^{\beta} \} \\
&\leqslant C'_{\delta,\rho} (A + \delta)^m (pm)!^{\alpha} (B + \rho)^q (pq)!^{\beta}
\end{aligned}$$

for every $m \in N$, $q \in N - \{0\}$ and $\delta, \rho > 0$ in virtue of the Stirling formula and the inequality $q < (1 + \epsilon)^q C_\epsilon$ where C_ϵ is a positive constant, for every $\epsilon > 0$.

Moreover $| x^m x^{-\mu} x \psi(x) | \leqslant C'_{\delta,\rho} (A + \delta)^m (pm)!^{\alpha} (B + \rho)^0 (0\,p)!^{\beta}$

for $m \in N$ and $\delta, \rho > 0$.

Hence, $x\psi$ is in ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ and the mapping is continuous.

The proof is completed by induction on n.

The procedure is analogous in any of the spaces under consideration.

Property 3.2—Let l be a real number. If we denote by ${}^{p}S^{\beta}_{\mu,\alpha}$ any of the spaces

${}^{p}S_{\mu,\alpha,A}$, ${}^{p}S_{\mu}^{\beta,B}$, ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ or the respective union spaces, then the operator

$$x^{l} : {}^{p}S_{\mu,\alpha}^{\beta} \to {}^{p}S_{\mu+l,\alpha}^{\beta}$$

is an isomorphism.

It can be easily proved by using the definitions of these spaces.

Property 3.3—The operator $P_{\mu} = x^{\mu+1} Dx^{-\mu}$ is an isomorphism of ${}^{p}S_{\mu,\alpha}^{\beta}$ onto ${}^{p}S_{\mu+1,\alpha}^{\beta}$, and its inverse is given by

$$P_{\mu}^{-1}(\psi)(x) = x^{\mu} \int_{\infty}^{x} t^{-\mu-1} \psi(t)\, dt.$$

PROOF : Operator P_{μ} and its inverse are linear. If $\psi \in {}^{p}S_{\mu,\alpha,A}^{\beta,B}$ then

$$| x^{m} D^{q} (x^{-\mu-1} (P_{\mu} \psi)(x) | \leqslant | x^{m} D^{q+1} (x^{-\mu} \psi(x)) |$$
$$\leqslant C_{\delta,\rho} (A+\delta)^{m} (pm)!^{\alpha} (B+\rho)^{q} (pq)!^{\beta}$$

for every $m, q \in N$ and $\delta, \rho > 0$. Here too it is enough to use Stirling formula.

Hence, P_{μ} is a continuous mapping.

Moreover if $\psi \in {}^{p}S_{\mu+1,\alpha,A}^{\beta,B}$, then:

$$| x^{m} D^{q} (x^{-\mu} P_{\mu}^{-1} \psi(x)) | = | x^{m} D^{q} \left(\int_{\infty}^{x} t^{-\mu-1} \psi(t)\, dt \right) | = L(x, m, q)$$

expression that, if $q > 0$, results equal or less than

$$| x^{m} D^{q-1} (x^{-\mu-1} \psi(x)) | \leqslant C_{\delta,\rho} (A+\delta)^{m} (pm)!^{\alpha} (B+\rho)^{q} (pq)!^{\beta}$$

If $q = 0$, one has:

$$L(x, m, 0) \leqslant \int_{0}^{\infty} \frac{(t^{m+2} + t^{m}) (t^{-\mu-1} | \psi(t) |)}{1+t^{2}}\, dt < C_{\delta,\rho} (A+\delta)^{m} (pm)!^{\beta}.$$

Hence $P_{\mu}^{-1} \psi$ is in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$, and P_{μ}^{-1} is a continuous operator.

The proof is similar in the case of spaces of the type ${}^{p}S_{\mu}^{\beta,B}$ or ${}^{p}S_{\mu,\alpha,A}$.

Property 3.4—The mapping $D : {}^{p}S^{\beta}_{\mu,\alpha} \rightarrow {}^{p}S^{\beta}_{\mu-1,\alpha}$ is linear and continuous.

PROOF : We limit to ourselves to the case of the operator

$$D : {}^{p}S^{\beta,B}_{\mu,\alpha,A} \rightarrow {}^{p}S^{\beta,B}_{\mu-1,\alpha,A}$$

since the other cases can be deduced from this one.

Let $\psi \in {}^{p}S^{\beta,B}_{\mu,\alpha,A}$. Then,

$$\begin{aligned} | x^m D^q (x^{-\mu+1} \psi(x)) | &\leqslant | x^m D^q (x^{-\mu} \psi(x)) (\mu + q) | \\ &\quad + | x^{m+1} D^{q+1} (x^{-\mu} \psi(x)) | \\ &\leqslant C_{\delta,\rho} (A + \delta)^m (pm)!^{\alpha} (B + \rho) (pq)!^{\beta} \end{aligned}$$

for every $m, q \in N$ and $\delta, \rho > 0$, following a procedure similar to the one used above.

From the previous results it can be easily inferred.

Property 3.5—The operator $B_\mu = DP_\mu$ from ${}^{p}S^{\beta}_{\mu,\alpha}$ into itself is linear and continuous.

Defining the generalized D^*, P^*_μ, P^{-1*}_μ and B^*_μ as the adjoint of the classical operators D, P_μ, P^{-1}_μ and B_μ respectively the following:

Property 3.6—The operators $D^* : \left({}^{p}S^{\beta}_{\mu-1,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ and $B^*_\mu : \left({}^{p}S^{\beta}_{\mu,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ are linear and continuous.

The mapping $P^*_\mu : \left({}^{p}S^{\beta}_{\mu+1,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ is an isomorphism P^{-1*}_μ is its inverse.

4. MULTIPLIERS IN SPACES OF TYPE ${}^{p}S^{\beta}_{\mu,\alpha}$

We are now interested in smooth functions on $0 < x < \infty$ which are multipliers in spaces of type ${}^{p}S^{\beta}_{\mu,\alpha}$.

Let $\theta \in C^\infty (I)$ be a function such that:

$$| D^q \theta(x) | \leqslant C B_0{}^q (pq)!^{\beta} (1 + x^l)$$

where $l \in N$ and $C, B_0 > 0$. Also, let $\psi \in {}^pS_{\mu,\alpha,A}$. Hence

$$| x^m D^n (x^{-\mu} \theta (x)\psi(x)) | \leqslant C_{n,\delta} \sum_{r=0}^{n} \binom{n}{r} B_0{}^r (pr)!^{\beta} (A + \delta)_m (pm)!^{\alpha}$$

$$\times (1 + (A + \delta) \frac{(p(m + l))!^{\alpha}}{(pm)!^{\alpha}}$$

$$< C'_{n,\delta} (A + \delta)^m (pm)!^{\alpha}$$

for $m, n \in N$ and $\delta > 0$.

Thus $\theta\psi$ is in ${}^pS_{\mu,\alpha,A}$ and the mapping

$${}^pS_{\mu,\alpha,A} \to {}^pS_{\mu,\alpha,A}$$

$\psi \to \theta\psi$ is continuous.

Moreover taking a ψ in ${}^pS_{\mu}^{\beta,B}$

$$| x^m D^n (x^{-\mu} \psi (x) \theta (x)) | \leqslant C_{m,\rho} (pm)!^{\beta} \sum_{r=0}^{n} \binom{n}{r} B_0{}^r (B + \rho)^{n-r}$$

$$= C_{m,\rho} (pm)!^{\beta} (B + B_0 + \rho)^n$$

for $m, n \in N$ and $\rho > 0$, since $(pr)! \cdot (p(n - r))! < (pn)!$. Therefore $\theta\psi$ is in ${}^pS_{\mu}^{\beta,B+B_0}$ and the operator

$${}^pS_{\mu}^{\beta,B} \to {}^pS_{\mu}^{\beta,B+B_0}$$

$\psi \to \theta\psi$ is continuous.

If $\psi \in {}^pS_{\mu,\alpha,A}^{\beta,B}$ then

$$| x^m D^n (x^{-\mu} \psi (x) \theta (x)) | < C_{\delta,\rho} (A + \rho)^m (pm)!^{\alpha} (pn)!^{\beta} (B + B_0 + \rho)^n$$

for every $m, n \in N$ and $\delta, \rho > 0$. Hence $\psi\theta$ is in ${}^pS_{\mu,\alpha,A}^{\beta,B+B_0}$, and the mapping

$${}^pS_{\mu,\alpha,A}^{\beta,B} \to {}^pS_{\mu,\alpha,A}^{\beta,B+B_0}$$

$\psi \to \theta\psi$ is continuous.

These facts are summarized in the following:

Property 4.1—If $\theta \in C^{\infty}(I)$ and $| D^q \theta (x) | \leqslant C B_0{}^q (pq)!^{\beta} (1 + x^l)$ for every $q \in N$, with $l \in N$ and $B_0, C > 0$, then θ is a multiplier of

(a) ${}^{p}S_{\mu,\alpha,A}$ into itself (and of $({}^{p}S_{\mu,\alpha,A})'$ into itself),

(b) ${}^{p}S_{\mu}^{\beta,B}$ into ${}^{p}S_{\mu}^{\beta \cdot B+B_0}$ $\left(\text{and of } \left({}^{p}S_{\mu}^{\beta \cdot B+B_0} \right)' \text{ into } \left({}^{p}S_{\mu}^{\beta,B} \right)' \right)$,

(c) ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ into ${}^{p}S_{\mu,\alpha,A}^{\beta \cdot B+B_0}$ $\left(\text{and of } \left({}^{p}S_{\mu,\alpha,A}^{\beta \cdot B+B_0} \right)' \text{ into } \left({}^{p}_{c}S_{\mu,\alpha,A}^{\beta,B} \right)' \right)$.

This result can be extended to the respective union spaces.

We now consider the set M, defined by Méndez[5], consisting of the smooth functions on $0 < x < \infty$ such that for each $r \in N$ there exists a $n_r \in N$ for which the function

$$\frac{D^r \theta(x)}{1 + x^{n_r}}$$

is bounded on $0 < x < \infty$. M is a space of multipliers in H_μ.

Property 4.2—If $\theta \in M$, the mapping

$${}^{p}S_{\mu,\alpha,A} \to {}^{p}S_{\mu,\alpha,A}$$

$\psi \to \psi\theta$ is linear and continuous.

PROOF : It is enough to check that

$$| x^m D^q (x^{-\mu} \theta(x) \psi(x)) | \leqslant \sum_{r=0}^{q} \binom{q}{r} \frac{| D^r \theta(x) |}{1 + x^{n_r}} (x^m + x^{m+n_r})$$

$$\times \; | D^{q-r} (x^{-\mu} \psi(x)) | \leqslant C_{q,\delta} (A+\delta)^m (pm)!^{\alpha}$$

for each $\psi \in {}^{p}S_{\mu,\alpha,A}$, $m, q \in N$, and suitable nonnegative integers n_r.

This result can be extended to the union space ${}^{p}S_{\mu,\alpha}^{\beta}$ and the respective dual spaces.

Note that a similar result cannot always be extended for ${}^{p}S_{\mu}^{\beta,B}$ and ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$.

5. THE HANKEL-CLIFFORD TRANSFORMATION IN THE SPACES OF TYPE ${}^{p}S_{\mu,\alpha}^{\beta}$

An integral transform given by the pair

$$F(y) = h_\mu \{f(x)\}(y) = y^\mu \int_0^\infty C_\mu (xy) f(x) \, dx$$

$$\ldots(1)$$

$$f(x) = h_\mu \{F(y)\}(x) = x^\mu \int_0^\infty C_\mu (xy) F(y) \, dy$$

was defined by Méndez[5], who called it the Hankel-Clifford transformation. The kernel of this transform, C_μ, is the Bessel-Clifford function of the first kind and order μ. C_μ is related to the Bessel function J_μ of the first kind by $C_\mu(z) = z^{-\mu/2} J_\mu(2\sqrt{z})$ (see Hayek[3]).

This transformation is an automorphism onto H_μ, for $\mu \geqslant 0$.

Since $\frac{d^n}{dz^n} C_\mu(z) = (-1)^n C_{\mu+n}(z)$, $n \in N$, then for every $\phi \in H_\mu$ and $m, k \in N$

$$y^m D^k y^{-\mu} \psi(y) = (-1)^k \int^{\infty} C_{\mu+k+2m}(xy)(xy)^m x^{\mu+k+m} D^{2m} x^{-\mu} \phi(x)\, dx$$

where $\psi(y) = h_\mu\{\phi(x)\}(y)$.

By dividing the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$ and in virtue of the boundedness of the function $z^m C_{\mu+k+2m}(z)$, we obtain

$$|y^m D^k (y^{-\mu} \psi(y))| \leqslant M \{\sup_{x \in I} |x^{c+k+m} D^{2m} x^{-\mu} \phi(x)| + \sup_{x \in I} |x^{c+k+m+2} D^{2m} x^{-\mu} \phi(x)|\} \quad \ldots(2)$$

for $m, k \in N$ and $\mu \geqslant 0$, being $c = [\mu]$ and M a constant.

We now study the image of ${}^pS^{\beta}_{\mu,\alpha}$ by h_μ. Recall that these spaces are contained in H_μ.

Let ϕ be an element of ${}_pS_{\mu,\alpha,A}$, invoking (2) we obtain

$$|y^m D^k (y^{-\mu} \psi(y))| \leqslant K_{m,\delta} (A+\delta)^k \{(p(m+k+c))!^\alpha + (p(c+k+m+2))!^\alpha\} \leqslant C_{m,\delta}(A+\delta)^k (pk)!^\alpha$$

for every $m, k \in N$ and $\delta > 0$.

Hence, the mapping $h_\mu : {}^pS_{\mu,\alpha,A} \rightarrow {}^pS^{\alpha,A}_{\mu}$ is linear and continuous.

If $\phi \in {}^p\hat{S}^{\beta,B}_{\mu}$, then

$$|y^m D^k y^{-\mu} \psi(y)| \leqslant M\{C_{k+m+c,\rho}(B+\rho)^{2m}(2mp)!^\beta + C_{k+m+2+c,\rho}(B+\rho)^{2m}(2mp)!^\beta\}.$$

Since $C_{k+m+c,\rho} < C'_{k,\rho}$ for every $m \in N$

$$|y^m D^k y^{-\mu} \psi(y)| \leqslant M_{k,\eta}(B^2+\eta)^m (2mp)!^{2\beta}$$

for $m, k \in N$ and $\eta > 0$.

Also, from the Stirling formula the next equality follows:

$$| y^m D^k y^{-\mu} \psi(y) | \leqslant M_{k,\eta} (B^2 2^{2p} + \eta)^m (pm)!^{2\beta}$$

for $m, k \in N$ and $\eta > 0$.

Therefore the mappings:

$$h_\mu : {}^p\hat{S}_\mu^{\beta,B} \rightarrow {}^{2p}S_{\mu,\beta,B^2}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta,B} \rightarrow {}^pS_{\mu,2\beta,2^{2p\beta}B^2}$$

are linear and continuous.

Let now ϕ be in ${}^pS_{\mu,\alpha,A}^{\beta,B}$. According to (2), we have

$$\begin{aligned} | y^m D^k y^{-\mu} \psi(y) | &\leqslant M_{\delta,\rho} \{(A + \delta)^{m+k+c} (p(c + k + m))!^\alpha (B + \rho)^{2m} \\ &\times (2pm)!^\beta + (A + \delta)^{c+k+m+2} (p(c+k+m+2))!^\alpha \\ &\times (B + \rho)!^{2m} (2pm)!^\beta\} \leqslant C_{\eta,\epsilon} (Ae^{p\alpha} + \eta)^k \\ &\times (pm)!^{2\beta+\alpha} (AB^2 2^{2p\beta} e^{p\alpha} + \epsilon)^m (pk)!^\alpha \end{aligned}$$

for every $m, k \in N$ and $\eta, \epsilon > 0$.

Thus, it has been established that the mapping

$$h_\mu : {}^pS_{\mu,\alpha,A}^{\beta,B} \rightarrow {}^pS_{\mu,2\beta+\alpha,B^2A2^{2p\beta}e^{p\alpha}}^{\alpha,Ae^{p\alpha}}$$

is linear and continuous.

The above results are summarized in the next

Theorem 5.1—The mappings

$$h_\mu : {}^pS_{\mu,\alpha,A}^{\beta,B} \rightarrow {}^pS_{\mu,2\beta+\alpha,AB^22^{2p\beta}e^{p\alpha}}^{\alpha,Ae^{p\alpha}}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta,B} \rightarrow {}^{2p}S_{\mu,\beta,B^2}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta,B} \rightarrow {}^pS_{\mu,2\beta,2^{2p\beta}B^2}$$

$$h_\mu : {}^pS_{\mu,\alpha,A} \rightarrow {}^pS_\mu^{\alpha,A}$$

are linear and continuous.

Defining the generalized transformation h'_μ as the adjoint of the classical transformation h_μ, it can easily be seen that

Theorem 5.2—The operators

$$h'_\mu : \left({}^{p}S^{\alpha, Ae^{p\alpha}}_{\mu, 2\beta+\alpha, AB^2 2_2 {}^{2p\beta}e\, p\alpha} \right)' \to \left({}^{p}S^{\beta, B}_{\mu, \alpha, A} \right)'$$

$$h'_\mu : ({}^{p}S_{\mu, 2\beta, 2\ {}^{2p\beta}B\ 2})' \to \left({}^{p}\hat{S}^{\beta, B}_{\mu} \right)'$$

$$h'_\mu : ({}^{2p}S_{\mu, \beta, B^2})' \to \left({}^{p}\hat{S}^{\beta, B}_{\mu} \right)'$$

$$h'_\mu : \left({}^{p}S^{\alpha, A}_{\mu} \right)' \to ({}^{p}S_{\mu, \alpha, A})'$$

are linear and continuous.

6. A Cauchy Problem Containing the Bessel-type Operator $B_\mu = Dx^{\mu+1} Dx^{-\mu}$

We consider the Cauchy problem

$$\left. \begin{aligned} \frac{\partial u(x, t)}{\partial t} &= P(B_\mu)\, u(x, t) \\ u(x, t_0) &= \phi_0(x) \end{aligned} \right\} \qquad \ldots(3)$$

for $0 \leqslant t \leqslant t_0 \leqslant T$, with $u(x, t)$ an unknown vector function and P a square matrix of polynomials. The initial value ϕ_0 is in a fundamental space that will be determined later.

Application of the h_μ-transform to problem (3) leads to

$$\left. \begin{aligned} \frac{\partial v(y, t)}{\partial t} &= P(-y)\, v(y, t) \\ v(y, t_0) &= \psi_0(y) \end{aligned} \right\} \qquad \ldots(4)$$

where $v(y, t) = h_\mu \{u(x, t)\,;\, x \to y\}$ and $\psi_0(y) = h_\mu \{\psi_0(x)\}(y)$. A formal solution of (4) is

$$v(y, t) = Q(-y, t, t_0)\, \psi_0(y)$$

where

$$Q(-y, t, t_0) = \exp((t - t_0)\, P(-y)).$$

A bound for the matrix $Q(s, t, t_0)$ is given by (see Geland and Shilov[1], pp. 53)

$$\| Q(s, t, t_0) \| \leqslant C \exp\left(\left(\frac{\theta}{2}\right)^{p_0} \frac{1}{p_0} |s|^{p_0} \right)$$

for $t \in [0, T]$ (even $t \in [0, T + t_0]$), $2^{p_0+1} p_0 T < \frac{1}{p_0} \theta^{p_0}$ and p_0 being the reduced order of the system (3) when the operator B_μ is substituted by $i \frac{\partial}{\partial x}$.

In virtue of Cauchy's integral formula

$$D^q Q(-y, t, t_0) = \frac{q}{2\pi i} \int_0^{2\pi} \frac{Q(-y + Re^{i\alpha}, t, t_0) Re^{i\alpha} i}{(Re^{i\alpha})^{q+1}} d\alpha.$$

Hence

$$\| D^q Q(-y, t, t_0) \| \leqslant q! R^{-q} C \exp\left(\frac{(2\theta)^{p_0}}{p_0} y^{p_0}\right) \exp\left(\frac{(2\theta)^{p_0} R^{p_0}}{p_0}\right)$$

where $C > 0$.

If we assume $R = q^{1/p_0}$, then

$$\| D^q Q(-y, t, t_0) \| \leqslant C_1 q^{q\left(a - \frac{1}{q_0}\right)} \exp\left(\frac{(2\theta)^{p_0}}{p_0} y^{p_0}\right) \times \left(\exp\left(\frac{(2\theta)^{p_0}}{p_0}\right)\right)^q$$

for $a \geqslant 1$.

And we arrive at the following

Property 6.1—The matrix $Q(-y, t, t_0)$ is a multiplier of the space

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right), B}_{\mu, \frac{1}{pp_0}, A} \quad \text{in} \quad {}^pS^{a - \frac{1}{p_0}, Bp^{\left(a - \frac{1}{p_0}\right)} + B_0}_{\mu, \frac{1}{p_0}, Ap'^{1/p_0} + A_0} \qquad \text{for } p > 1$$

where

$$B_0 = \exp((2\theta)^{p_0}/p_0)$$

$$A_0 = \left(\frac{1}{pp_0\, e\, (c - a_0)}\right)^\alpha$$

with

$$a_0 = \frac{(2\theta)^{p_0}}{p_0} \quad c = \frac{\alpha}{e} A^{-1/\alpha} \quad \alpha = \frac{1}{pp_0}.$$

If $p = 1$ then $Q(-y, t, t_0)$ is a multiplier of

$${}^1S^{a - \frac{1}{p_0}, B}_{\mu, \frac{1}{p_0}, A} \quad \text{in} \quad {}^1S^{\epsilon\left(a - \frac{1}{p_0}\right), B + B_0}_{\mu, \frac{\epsilon}{p}, A + A_0}$$

with $\epsilon > 1$.

This property has sense provided that the space ${}^{p}S^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ under consideration is nontrivial.

By invoking 1.5 we get

Property 6.2—The space ${}^{p}S^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ is nontrivial if

(a) $p_0 > \dfrac{1}{a}$ and $a > p \geq 1$

(b) $p_0 > \dfrac{1}{a}$, $a = p > 1$ and $A \cdot B > \gamma$, γ being a positive constant

(c) $ap_0 = 1$ and $\dfrac{1}{pp_0} > 1$.

We now define the fundamental space to ϕ_0 (the data of the initial value problem (3)) belongs to. According to Theorem 1, if $\Phi = {}^{p}S^{\beta,B'}_{\mu,\alpha,A'}$ and

$$h_\mu\{\Phi\} \subseteq {}^{p}S^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$$

(provided that ${}^{p}S^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ is nontrivial) then:

$$\beta = \frac{1}{pp_0} - \frac{a}{2p}, \quad \alpha = \frac{1}{p}\left(a - \frac{1}{p_0}\right) \text{ and } A', B' > 0 \text{ with}$$

$$A'\,B'\,2^{2p\beta}\,e^{p\alpha} = A, \quad A'\,e^{p\alpha} = B.$$

The space Φ is nontrivial if some of these conditions hold

(a) $\dfrac{a}{2p} > 1$ and $a > \dfrac{1}{p_0} > \dfrac{a}{2}$

(b) $a = 2p$, $a > \dfrac{1}{p_0} > \dfrac{a}{2}$ and $A'\,B' > \gamma$, where γ is a positive constant

(c) Either $a = \dfrac{1}{p}$ and $1 > 2pp_0$, or $ap_0 = 2$ and $1 > pp_0$.

When the space Φ is trivial a new

$$\Phi = \{\psi \in H_\mu : {}_c h_\mu \, \psi \in {}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right),\, B}_{\mu,\, \frac{1}{pp_0},\, A}\}$$

has to be introduced.

In this case Φ is endowed with the graph topology. This topology is generated by the multinorm $\left\{ | \ | {}^{\rho}_{\delta} \right\}_{\delta, \rho > 0}$ where

$$| \psi | {}^{\rho}_{\delta} = \| {}_c h_\mu \, \psi \| {}^{\rho}_{\delta} \quad \text{for } \psi \in \Phi \text{ and } \delta, \rho > 0$$

and where the last seminorm is defined in the space

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right),\, B}_{\mu,\, \frac{1}{pp_0},\, A}.$$

The nontriviality of Φ is deduced from the nontriviality of

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right),\, B}_{\mu,\, \frac{1}{pp_0},\, A}.$$

The following diagram summarizes our last results

$$\Phi \rightarrow \Phi_1 = {}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right),\, B}_{\mu,\, \frac{1}{pp_0},\, A} \rightarrow \Phi_2 = {}^pS^{a - \frac{1}{p_0},\, Bp^{p\beta} + B_0}_{\mu,\, \frac{1}{p_0},\, Ap^{p\alpha} + A_0}$$

$$\phi_0 \rightarrow h_\mu \{\phi_0\} = \psi_0 \rightarrow v(y \ t) = Q(-y, t, t_0) \, \psi_0(y).$$

Note that Φ_1 is contained in Φ_2 and the inclusion is continuous.

We now show that $v(y, t)$ is the solution of the problem (3).

Property 6.3—If $\Phi_0 \in \Phi$, then

$\lim\limits_{t \to t_0} v(y,t) = \psi_0(y)$ and $\lim\limits_{\Delta t \to 0} \dfrac{\Delta v(y, t)}{\Delta t} = P(-y) \, v(y, t)$ in the sense of the convergence in Φ_2.

Hence, in virtue of the operational rules in Méndez[5] and taking into account that ${}_c h_\mu$ is a continuous mapping, we get

$$\lim_{t \to t_0} h_\mu \{v(y, t), y \to x\} = h_\mu \{\psi_0(y)\}(x) = \phi_0(x)$$

$$\lim_{\Delta t \to 0} \frac{\Delta \, h_\mu \{v(y, t), y \to x\}}{\Delta t} = P(B_\mu) \, h_\mu \{v(y, t), y \to x\}$$

in the sense of convergence in the space

$$\Phi_4 = {}_pS^{\frac{1}{p_0},\ (Ap^{\nu\alpha} + A_0)\, e^{p/p_0}}_{\mu,\ 2a - \frac{1}{p_0},\ (Ap^{\nu\alpha} + A_0)\,(Bp^{\nu\beta} + B_0)^2\, 2^{2p\left(a - \frac{1}{p_0}\right)} e^{p/p_0}}.$$

Thus, the next result follows.

Theorem 6.1—The function $u(x, t) = h_\mu \{Q(-y, t, t_0)\, h_\mu \{\phi_0(x)\}(y), y \to x\}$ in Φ_4 is a solution of the Cauchy problem (3), for every initial value $\phi_0 \in \Phi$.

7. Existence of Generalized Solutions

We now consider the initial value problem

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= P(B_\mu^*)\, u(x, t) \\ u(x, 0) &= u_0(x) \text{ with } u_0 \in \Phi_4' \end{aligned} \qquad \ldots(5)$$

The generalized Hankel-Clifford transform h'_μ, leads to the new equivalent problem

$$\begin{aligned} \frac{\partial v(y, t)}{\partial t} &= P(-y)\, v(y, t) \\ v(y, 0) &= v_0(y) \end{aligned} \qquad \ldots(6)$$

where $v(y, t) = h'_\mu \{u(x, t), x \to y\}$ and $v_0(y) = h'_\mu \{u_0(x)\}(t)$.

A formal solution of (6) is the generalized function $v(y, t) = v_0(y)\, e^{-vt}$.

The distribution $u(x, t) = h'_\mu \{v_0(y)\, e^{-vt}, y \to x\} \in \Phi'$ is a solution of (5).

In effect, according to Theorem 6.1 one has:

(a) $$\begin{aligned} \frac{\partial}{\partial t} < h'_\mu \{e^{-vt} v_0(y)\}, \phi > &= \frac{\partial}{\partial t} < u_0, h_\mu \{e^{-vt} h_\mu \{\phi\}\} > \\ &= < u_0, h_\mu \{e^{-vt} h_\mu \{P(B_\mu)\}\} > \\ &= < P\left(B_\mu^*\right) h'_\mu\, e^{-yt} h'_\mu\, u_0, \phi > \end{aligned}$$

for every $\phi \in \Phi$ and

(b) $< h'_\mu \{e^{-vt} h'_\mu \{u_0\}\}, \phi > = < u_0, h_\mu \{e^{-vt} h_\mu \{\phi\}\} > \to < u_0, \phi >$

for every $\phi \in \Phi$.

Thus, we arrive at the following

Theorem 7.1—The generalized function

$$u\ (x,\ t) = h'_{\mu}\ \{e^{-yt}\ h'_{\mu}\ \{u_0\}\} \in \Phi'$$

is solution of (5) for every initial value $u_0 \in \Phi'_4$.

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A FINITE INTEGRAL INVOLVING A GENERAL CLASS OF POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

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In this paper, we evaluate a finite integral involving the product of a general class of polynomials and the multivariable *H*-function. On account of the most general nature of the polynomials and multivariable *H*-function, our result provide interesting unifications and extensions of a number of (known and new) integrals. Integrals obtained by Gupta *et al.*[2] (p. 69), Garg[1] (p. 251) and many other integrals follow as particular cases of our main result.

1. INTRODUCTION

The multivariable *H*-function has been defined by Srivastava and Panda[6]. We shall use the following contracted form Srivastava *et al.*[5] [p. 251 eqn. (C. 1)]:

$$H[z_1, \ldots, z_r] = H^{0,\, N\, :\, M',\, N';\, \ldots;\, M^{(r)},\, N^{(r)}}_{P,\, Q\, :\, P',\, Q';\, \ldots;\, P^{(r)},\, Q^{(r)}} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \;\middle|\; \begin{matrix} \left(a_j;\, \alpha'_j, \ldots, \alpha^{(r)}_j\right)_{1,P} : \left(c'_j, \gamma'_j\right)_{1,P'}; \ldots; \left(c^{(r)}_j, \gamma^{(r)}_j\right)_{1,P^{(r)}} \\ \\ \left(b_j;\, \beta'_j, \ldots, \beta^{(r)}_j\right)_{1,Q} : \left(d'_j, \delta'_j\right)_{1,Q'}; \ldots; \left(d^{(r)}_j, \delta^{(r)}_j\right)_{1,Q^{(r)}} \end{matrix} \right] \quad \ldots(1.1)$$

to denote the *H*-function of r complex variables $z_1, \ldots, z_r$. All the Greek letters are assumed to be positive real numbers for standardization purposes; the definition of the multivariable *H*-function will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in the paper referred to above.

Srivastava[3] introduced the general class of polynomials (see also Srivastava[4] and Srivastava and Singh[7]) :

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k,\; n = 0, 1, 2, \ldots \qquad \ldots(1.2)$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ are arbitrary constants real or complex. By suitably specializing the coefficients $A_{n,k}$, the general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above.

2. Result Required

The following result given by Garg[1] (p. 244) will be required in establishing our main integral in the next section :

$$\int_0^a x^{\rho-1} (a + x)^{\sigma} (c + bx)^{-\lambda}\, H\, [z_1\, x^{u_1} (a - x)^{v_1} (c + bx)^{-w_1}, \ldots,$$

$$z_r\, x^{u_r} (a - x)^{v_r} (c + bx)^{-w_r}]\, dx$$

$$= \sum_{l=0}^{\infty} \frac{(-ab)^l\, a^{\rho+\sigma}}{c^{\lambda+l}\, l!}\, H^{0, N+3\; :\; *}_{P+3, Q+2\; :\; *} \left[\begin{array}{c|c} z_1\, c^{-w_1}\, a^{u_1+v_1} & A : * \\ \vdots & \\ z_1\, c^{-w_r}\, a^{u_r+v_r} & B : * \end{array} \right] \qquad \ldots(2.1)$$

where $H[z_1, \ldots, z_r]$ denotes the multivariable H-function defined by (1.1). The asterisk ($*$) occurring in (2.1) indicates that parameters at those places are the same as the parameters of the multivariable H-function in (1.1) at corresponding places. This notation will be adhered to throughout this paper. Also

$$A = (-\sigma; v_1, \ldots, v_r),\; (1 - \rho - l; u_1, \ldots, u_r),\; (1 - \lambda - l; w_1, \ldots, w_r),$$

$$\left(a_j; \alpha_j', \ldots, \alpha_j^{(r)} \right)_{1,P}$$

$$B = \left(b_j; \beta_j', \ldots, \beta_j^{(r)} \right)_{1,Q},\; (1 - \lambda; w_1, \ldots, w_r),$$

$$(-\rho - \sigma - l; u_1 + v_1, \ldots, u_r + v_r).$$

The conditions of validity of (2.1) can be found in the reference given above.

3. Main Integral

We shall establish the following finite integral in this section :

$$\int_0^a x^{\rho-1} (a - x)^{\sigma} (c + bx)^{-\lambda}\, S_n^m\, [\, yx^u (a - x)^v (c + bx)^{w}]$$

$$\times\; S_{n'}^{m'}\, [zx^{u'} (a - x)^{v'} (c + bx)^{w'}]$$

(equation continued on p. 606)

$$\times H[z_1 x^{u_1}(a-x)^{v_1}(c+bx)^{-w_1}, \ldots, z_r x^{u_r}(a-x)^{v_r}$$
$$\times (c+bx)^{-w_r}]\, dx$$

$$= \sum_{l=0}^{\infty} \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-1)^l (-n)_{mk} (-n')_{m'k'} y^k z^{k'} a^{\rho+\sigma+k(u+v)+k'(u'+v')+l} b^l}{l!\, k!\, k'!\, c^{\lambda+wk+w'k'+l}}$$

$$\times A_{n,k} A'_{n',k'} H^{0,N+3\,:\,*}_{P+3,Q+2\,:\,*}\left[\begin{array}{l|l} z_1 c^{-w_1} a^{u_1+v_1} & C:* \\ \vdots & \\ z^r c^{-w_r} a^{u_r+v_r} & D:* \end{array}\right] \quad \ldots (3.1)$$

where

$$C = (-\sigma - vk - v'k'; v_1, \ldots, v_r), (1-\rho-l-uk-u'k; u_1, \ldots, u_r)$$
$$(1-\lambda-l-wk-w'k'; w_1, \ldots, w_r), \left(a_j; \alpha'_j, \ldots, \alpha_j^{(r)}\right)_{1,P}$$

$$D = \left(b_j; \beta'_j, \ldots, \beta_j^{(r)}\right)_{1,Q}, (1-\lambda-wk-w'k'; w_1, \ldots, w_r)$$
$$(-\rho-\sigma-l-(u+v)k-(u'+v')k'; u_1+v_1, \ldots, u_r+v_r)$$

m and m' are arbitrary positive integers and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ and $A'_{n',k'}$ $(n', k' > 0)$ are arbitrary constants real or complex.

The (sufficient) conditions of validity of (3.1) are :

(i) a, b, c are positive numbers such that $\left|\frac{ab}{c}\right| < 1$

$\text{Re}(\lambda) > 0$, $\min\limits_{1 \leqslant i \leqslant r} (u, v, w, u', v', w', u_i, v_i, w_i) \geqslant 0$ (not all zero simultaneously)

(ii) $$\text{Re}(\rho) + \sum_{i=1}^{r} u_i \min_{1 \leqslant j \leqslant M} (i) \left\{ \text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) \right\} > 0$$

$$\text{Re}(\sigma) + \sum_{i=1}^{r} v_i \min_{1 \leqslant j \leqslant M} (i) \left\{ \text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) \right\} + 1 > 0$$

(iii) $\Omega_i > 0$, $|\arg z_i| < \frac{1}{2}\Omega_i \pi$, $\forall\, i \in \{1, 2, \ldots, r\}$, where

$$\Omega_i = -\sum_{j=N+1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{M^{(i)}} \delta_j^{(i)} - \sum_{j=M^{(i)}+1}^{Q^{(i)}} \delta_j^{(i)}$$

(*equation continued on p.* 607)

$$+ \sum_{j=1}^{N^{(i)}} \gamma_j^{(i)} - \sum_{j=N^{(i)}+1}^{P^{(i)}} \gamma_j^{(i)}.$$

(iv) The series occurring on right-hand side of (3.1) is absolutely convergent.

PROOF : To evaluate the integral (3.1), we first express both the general class of polynomials occurring in the integrand of (3.1) in series from given by (1.2) and then interchange the orders of summations and integration (which is permissible under the conditions stated above) so that the left-hand side of the integral (3.1) (say Δ) assumes the following form:

$$\Delta = \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-n)_{mk} (-n')_{m'k'} y^k z^{k'}}{k!\, k'!} A_{n,k} A'_{n',k'} \int_0^a x^{\rho+uk+u'k'-1}$$

$$\times (a - x)^{\sigma+vk+v'k'} (c + bx)^{-\lambda-wk-w'k'} H[z_1 x^{u_1} (a - x)^{v_1}$$

$$\times (c + bx)^{-w_1}, \ldots, z_r x^{u_r} (a - x)^{v_r} (c + bx)^{-w_r}]\, dx. \qquad \ldots(3.2)$$

Now, we evaluate the x-integral occurring in the above equation, with the help of (2.1) and arrive at the desired result (3.1) after a little simplification.

4. SPECIAL CASES AND APPLICATIONS

If we take $n' = 0$ (the polynomial $S_0^{m'}[x]$ will reduce to 1), $\lambda = w = 0$, $w_i = 0$ $(i = 1, 2, \ldots, r)$, $b \to 0$ and replace ρ by $\rho + 1$ in (3.1), we get a known integral obtained by Gupta *et al.*[2] [p. 69, eqn. (3.1)]. Further, if we take $u = 0$ and $v = 1$ in the integral so obtained, we get an integral given by Srivastava and Singh[7] [p. 166, eqn. (2.2), with $\xi = 0$].

Again, if we take $n' = 0$, $a = u = y = 1$, $v = w = 0$, $v_i = 0$ $(i = 1, 2, \ldots, r)$, $m = 1$, $A_{n,k} = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k}$ $\Big($ in this case $S_n^1[x] \to P_n^{(\alpha,\beta)}(1-2x)\Big)$ and replace σ by β in (3.1), we get another known integral given by Garg[1] [p. 251, eqn. (5.3.6)].

The importance of our main integral lies in its manifold generality. At the outset, we recall that in view of the generality of the polynomials $S_n^m[x]$, on suitably specializing the coefficients $A_{n,k}\, A'_{n',k'}$ and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh[7], our main integral can be reduced to a large number of integrals involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel poly-

nomials, Gould-Hopper polynomials, Brafman polynomials, and their various combinations.

Secondly, by specializing the various parameters and variables in the multivariable H-function, we can obtain from our main result, several integrals involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H-functions of one and several variables. Thus our main integral would at once yield a very large number of integrals involving a large variety of polynomials and various special functions occurring in the problems of mathematical analysis, applied mathematics, and mathematical physics.

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ON STRONGLY RARE-CONTINUITY

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A new weakly form of continuity which is weaker than weak-continuity and stronger than rare-continuity is introduced under the title of strongly-rare-continuity. Some certain characteristics of strongly rarely continuous functions, some of their properties in terms of α-topologies of Njastad and some of their relations with the other weakly types of functions are established.

Introduction

Many weak forms of continuity of single valued functions between two topological spaces has been defined in the literature since 1920's. Quasi-continuities of Blumberg[3] and Kempisty[8] were introduced in 1922 and 1932 in two different ways. Fomin[5] have defined the θ-continuity in 1941 and Levine have defined weak-continuity [9] and semi-continuity[10] in 1961 and 1963 respectively. Two independent kinds of almost continuity were defined by Husain[7] in 1966 and by Singal and Singal[19] in 1968. Faint-continuity of Long and Herrington[11], rare-continuity of Popa[16] and subweak-continuity of Rose[17] have been defined in the last decade for instance δ-continuity of Noiri which is an independent concept of continuity have also been defined in this period. The following non reversible implications among some of them are well known : continuity (resp. δ-continuity) $\Rightarrow$ almost continuity S & S $\Rightarrow$ θ-continuity $\Rightarrow$ weak-continuity $\Rightarrow$ rare-continuity. These are indeed weaker on independent forms of continuity. For example any function from any topological space to any hyperconnected space[20], i. e. the space in which all nonempty open subsets are dense is δ-continuous.

We define in this paper a new weak form of continuity under the title of strongly rare-continuity which is weaker than weak-continuity and stronger than rare-continuity. We determine some of its characteristics and some of its relations with the others. Some certain properties of strongly rarely continuous functions in terms of α-topologies of Njastad are also established.

Preparations

No specific separation axiom is assumed unless otherwise is explicitly stated. $\mathcal{N}x$ will denote the family of whole basic neighbourhoods of the point x in the space X. cl

$A = \bar{A}$, int $A = \mathring{A}$ and ∂A denote respectively the closure, the interior and the boundary of the subset A in X. A function $f : X \to Y$ is called weakly continuous (resp. θ-continuous, resp. δ-continuous, resp. almost continuous H., resp. almost continuous S and S, resp. rarely continuous) at $x \in X$ iff for each $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ such that $f(G_x) \subseteq \overline{W}_{f(x)}$ (resp $f(\overline{G}_x) \subseteq \overline{W}_{f(x)}$, resp. f (int $\overline{G}_x) \subseteq$ int $\overline{W}_{f(x)}$, resp. $G_x \subseteq$ cl $f^{-1}(W_{f(x)})$, resp. $f(G_x) \subseteq$ int $\overline{W}_{f(x)}$, resp. int $f(G_x) \subseteq \overline{W}_{f(x)}$) holds. The alpha and beta operations are defined as $\alpha\ A =$ int $\bar{A}$ and $\beta\ A =$ cl $\mathring{A}$ after Bourbaki[4]. As usual a subset A will be called nowhere dense iff $\alpha\ A = \phi$. We write $x \in A^*$ for a subset $A \subseteq X$ iff each $G_x \in \mathcal{N}_x$ is intersecting A with a non nowhere dense intersection, i. e. int $\overline{G_x \cap A} \neq \phi$. Then A^* is always closed, $\bar{A}^* = A^* \subseteq \bar{A}$ and $(A \cup B)^* = A^* \cup B^*$. Notice also that $\alpha A \subseteq A^*$ and therefore $\overline{\alpha A} \subseteq A^*$ hold since if a $U_x \in \mathcal{N}_x$ satisfy $U_x \subseteq \bar{A}$ then $\overline{G_x \cap A} \supseteq$ cl $(G_x \cap \overline{U_x \cap A}) \supseteq G_x \cap U_x$ could be written for any $G_x \in \mathcal{N}_x$. Since $\alpha\ (G \cap A) \overset{1}{=} \alpha(G \cap \alpha A)$ holds for any subset A and for any open G, one easily obtain $A^* \subseteq \overline{\alpha A}$. Thus $A^* = \overline{\alpha A}$ is found for any A. Notice that A is nowhere dense iff $A^* = \phi$ iff A^* is nowhere dense. The equality $(G \cap A^*)^* \overset{2}{=} (G \cap A)^{**}$ could also be proved for any open G and for any A. Hence the following equivalences hold for any open G and for any A : $G \cap \alpha A = \phi$ iff $G \cap A^* = \phi$ iff $G \cap A^*$ is nowhere dense iff $G \cap A$ is nowhere dense. The equalities (1) and (2) and the last equivalencies are also valid for any semi-open subset G.

$k_* A = A \cup A^*$ is therefore a closure operation on X by a theorem of Andrijevic[1] and even $k_* A = \mathrm{cl}_\alpha A$ holds[1] for each $A \subseteq X$. Here $\mathrm{cl}_\alpha A$ denotes the closure of A in the τ_α topology of Njastad[15], i. e. the topolgy on X which is precisely the family of those sets written as the difference of an open set and a nowhere dense set of X.

Since we use the τ_α topology in some frequency throughout the paper, we give here a short and independent proof that the family τ_α is indeed a topology on X. Notice that if G_μ is open and N_μ is nowhere dense in X for each index μ then

$$G_\mu - N_\mu \subseteq \alpha\ G_\mu = \text{int}\ \beta\ (G_\mu - N_\mu) \subseteq \text{int}\ \beta\ (\bigcup_\mu\ (G_\mu - N_\mu))$$

and thus

$$\bigcup_\mu\ (G_\mu - N_\mu) = \text{int}\ \beta\ (\mu\ (G_\mu - N_\mu)) - N_0$$

is found where the nowhere dense N_0 is defined as

$$N_0 = \text{int}\ \beta\ (\bigcup_\mu\ (G_\mu - N_\mu)) - \bigcup_\mu\ (G_\mu - N_\mu).$$

Njastad proved that $\tau \subseteq \tau_\alpha$ and the nowhere dense subsets of these topologies are the same and therefore note that $A^*(\tau) = A^*(\tau_\alpha)$ for each $A \subseteq X$[15]. The space on the set X equipped with the τ_α topology will briefly be denoted in this paper by X_α and a subset is called α-closed iff it is closed in X_α. The substar set of exclusively an inverse

set under any function $f: X \rightarrow Y$ is defined as follows : $x \in (f^{-1}(B))^*$ iff $(f(G_x) \cap B)^* = \phi$ for each $G_x \in \mathscr{N}_x$. Then all substar sets are closed,

$$(f^{-1}(B))^* \cup (f^{-1}(B))_* \subseteq \mathrm{cl}\, f^{-1}(B)$$

$$(f^{-1}(B \cup C))_* = (f^{-1}(B))_* \cup (f^{-1}(C))_*$$

$$(f^{-1}(B))_* = \phi \text{ if } B^* = \phi$$

and so $(f^{-1}(\bar{U}))_* = (f^{-1}(U))_*$ for each open $U \subseteq Y$. If the domain of f is compact then the equivalency $(f^{-1}(\bar{U}))_* = \phi$ iff $(U \cap f(X))_* = \phi$ holds for each open $U \subseteq Y$. The locally thinly scattered points of a function $f: X \rightarrow Y$ will be written by N_f and defined as follows : $x \in N_f$ iff $\exists\ G_x \in \mathscr{N}_x$; $(f(G_x))^* = \phi$. It is obvious that N_f is always open (may be empty) and satisfy $N_f \cap (f^{-1}(B))_* = \phi$ for all $B \subseteq Y$ E^1 and E^1_u will denote the one dimensional Euclidean space and the one dimensional upper limit topology of Sorgenfrey on reals respectively. We define simple and step functions just as in Royden[18]. In particularly the function $f(x) = [x] = \sup((-\infty, x] \cap Z)$ for each real x will be called as the classical step function where Z denotes the set of all integers. Semi-open (resp. regularly closed) subsets are those sets satisfying $A \subseteq \beta A$ (resp. $A = \beta A$) or equivalently those sets satisfying $G \subseteq A \subseteq \bar{G}$ (resp. $A = \bar{G}$) where G is an open set. H-closed spaces are those T_2 spaces which all their open coverings admit a finite and dense subfamilies or equivalently all their embeddings into Hausdorff spaces are closed[21]. Almost compact spaces are those which all their open coverings admit a finite subfamily such that the closures of members is a covering. Therefore H-closed spaces are precisely the almost compact T_2 spaces.

Strongly Rarely Continuous Functions

Theorem 1—The following are equivalent for any $f: X \rightarrow Y$.

(1) For each $W_{f(x)} \in \mathscr{N}_{f(x)}$ there exists a $G_x \in \mathscr{N}_x$ with $\alpha f(G_x) \subseteq \bar{W}_{f(x)}$ or equivalently $\alpha f(G_x) \subseteq \alpha W_{f(x)}$.

(2) For each $W_{f(x)} \in \mathscr{N}_{f(x)}$ there exists a $G_x \in \mathscr{N}_x$ such that $f(G_x) - W_{f(x)}$ or equivalently $f(G_x) - \bar{W}_{f(x)}$ is nowhere dense.

(3) For each $W_{f(x)} \in \mathscr{N}_{f(x)}$ there exists a $G_x \in \mathscr{N}_x$ and a nowhere dense $N_W \subseteq Y$ with $f(G_x) \subseteq \bar{W}_{f(x)} \cup N_W$ or equivalently $f(G_x) \subseteq W_{f(x)} \cup N_W$.

(4) For each $W_{f(x)} \subseteq \mathscr{N}_{f(x)}$ there exists a $G_x \in \mathscr{N}_x$ with $(f(G_x))^* \subseteq \bar{W}_{f(x)}$.

Proof : (1) $\Rightarrow$ (2) Easy since $\alpha(f(G_x) - W_{f(x)}) \subseteq \alpha f(G_x) - \bar{W}_{f(x)}$. Furthermore one can always write that

$$f(G_x) - W_{f(x)} = (f(G_x) - \bar{W}_{f(x)}) \cup (f(G_x) \cap \partial W_{f(x)})$$

where the second set participating the union is always nowhere dense.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are straightforward implications.

(3) ⇒ (4) Easy by taking the stars of the both sides of the inclusion relation of the hypothesis of (3).

Definition 1—A function f satisfying one of the conditions of Theorem 1 is called strongly rarely continuous at $x \in X$. Global definition of strongly rare-continuity could be given easily and expectedly. From now on we shortly write SRC for this type of functions.

Remark 1 : A function f is SRC (resp. rarely continuous) at $x \in X$ iff for each given $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ such that the part of the image of G_x scattering to the outside of $W_{f(x)}$ is nowhere dense (resp. rare, i. e. has an empty interior). Hence any function with the nowhere dense (resp. rare) image such as simple and step function is evidently SRC (resp. rarely continuous). Not all the SRC functions have nowhere dense images. Here follows an example : Let Q be the set of all rationals and P be the set of all irrationals and let the space X on reals be defined so that all irrational singletons are open and the basic neighbourhoods of any $x \in Q$ are $U_x(\epsilon) =]x - \epsilon, x + \epsilon[\cap Q$. Then the function f defined as $f(x) = [x] \chi_Q(x) + x.\chi_P(x)$ from X into E^1 is SRC. It is not weakly continuous on integers and $f(X)$ is not nowhere dense in E^1. Here χ_A denotes the characteristic function of A.

Remark 2 : One of the recent weaker form of weak-continuity has defined by Rose[17] as follows : $f : X \to Y$ is called subweakly continuous iff there is an open basis $\mathcal{B}$ of the topology on Y such that $\mathrm{cl}\, f^{-1}(B) \subseteq f^{-1}(\bar{B})$ for each $B \in \mathcal{B}$. Rose proved that[17] holding of the inclusions $\mathrm{cl}\, f^{-1}(U) \subseteq f^{-1}(\bar{U})$ for each open $U \subseteq Y$ is the necessary and sufficient condition for the weak-continuity of f. Subweak-continuity, weak-continuity and θ-continuity of an almost continuous H. Function are equivalent by the Theorem 6 and Theorem 10 of Rose[17]. Notice that the identic function from the cofinite topology on an infinite set X onto the discrete topology on X is subweakly continuous but not SRC. The classical step function from E^1 to E^1 is SRC but not subweakly continuous hence not weakly continuous. Thus subweak-continuity (resp. weak-continuity) and strongly rare continuity are independent of each other. Notice that the function $g : E^1 \to E^1_u$ defined as $g(x) = x.\chi_Q(x)$ is rarely continuous but not s. rarely continuous where Q denotes the set of all rationals in above. Therefore the evident implications

weak-continuity ⇒ strongly rare-continuity ⇒ rare-continuity are not reversible. Strongly rare-continuity is also different with an another recent and weaker form of weak-continuity defined by Long and Herrington[11] under the title of faint-continuity. The function defined in Example 2 of Long and Herrington[11] is faintly continuous but not SRC and the classical step function is not faintly continuous since it is easy to prove that $\tau = \tau_\theta^{11}$ iff τ is a regular topology, i. e. continuity and faint-continuity are coincided for any function defined into a regular space.

Theorem 2—Let the function $f : X \to Y$ be defined. Then

(i) f is strongly rarely continuous at $x \in X$ iff $x \notin (f^{-1}(Y - \overline{W}_{f(x)}))_*$ for each $W_{f(x)} \in \mathcal{N}_{f(x)}$.

(ii) The set of all points of which f is not strongly rarely continuous is

$$\cup\, [(f^{-1}(U))_* - f^{-1}(\bar{U}) : U \subseteq Y \text{ open}].$$

(iii) f is s. rarely continuous iff $(f^{-1}(U))_* \subseteq f^{-1}(\bar{U})$ for all open (or semi-open) $U \subseteq Y$.

(iv) f is strongly continuous iff $f : X \to Y_\alpha$ is Strongly rarely continuous.

(v) f is strongly rarely continuous iff all the restrictions $f \mid G_\mu$ (resp. $f \mid K_\mu$) are so where $(G_\mu)_\mu$ (resp. $(K_\mu)_\mu$) is an open (resp. locally finite closed) covering of X.

(vi) f is strongly rarely continuous iff f is so on an appropriate basic neighbourhoods of each point.

PROOF : (i) Let f be strongly rarely continuous at $x \in X$ and $W_{f(x)} \in \mathcal{N}_{f(x)}$ be given. Then there exists a $G_x \in \mathcal{N}_x$ with $(f(G_x))^* - \overline{W}_{f(x)} = \phi$ and so $(f(G_x) \cap (Y - \overline{W}_{f(x)}))^* = \phi$ i. e. $x \notin (f^{-1}(Y - \overline{W}_{f(x)})^*$ is found. Sufficiency can be proved just reversely.

(ii) If f is not strongly rarely continuous at $x \in X$ then one gets

$$x \in (f^{-1}(Y - \overline{W}_{f(x)}))_* - f^{-1}(\mathrm{cl}\,(Y - \overline{W}_{f(x)})).$$

If conversely $x \in (f^{-1}(U))_* - f^{-1}(\bar{U})$ for an open $U \subseteq Y$ then there exists a $W_{f(x)} \in \mathcal{N}_{f(x)}$ with $\overline{W}_{f(x)} \cap U = \phi$ and so the supposition of Strongly rare-continuity at $x \in X$ yields the existence of a $G_x \in \mathcal{N}_x$ with $(f(G_x) \cap U)^* = \phi$ a contradictory result with $x \in (f^{-1}(U))_*$.

(iii) Let f be strongly continuous and $U \subseteq Y$ be semi-open. Then

$$(f^{-1}(U))_* = (f^{-1}(W))_* \cup (f^{-1}(U - W))_* \subseteq f^{-1}(\overline{W}) = f^{-1}(\bar{U})$$

where the open W satisfy $W \subseteq U \subseteq \overline{W}$ and so $U - W \subseteq \partial W$ holds. Sufficiency is clear after the previous item (ii),

(iv) Notice firstly that if U is open and $N^* = \phi$ in the space Y then $\mathrm{cl}_\alpha\,(U - N) = \bar{U}$ holds by noticing $(U - N)^* = \overline{\alpha U - \beta N} = \bar{U}$. Hence the statement follows from the condition (2) of Theorem 1 by the equivalency of being nowhere dense in the spaces Y and Y_α.

(v) After the Theorem (3i) only the sufficiency of the second statement will be proved. Let all the restrictions $f \mid K_\mu$ be Strongly rarely continuous and $W_{f(x)} \in \mathcal{N}_{f(x)}$ be given. Then there exist the indexes $\mu_1, \mu_2, \ldots \mu_n$ and a $G_x \in \mathcal{N}_x$ such that

$$x \in \bigcap_{k \leqslant n} K_{\mu k},\ G_x \cap \bigcup_{\substack{\mu \neq \mu_k \\ k \leqslant n}} = \phi.$$

There also exist nowhere dense subsets $N_W^k \subseteq Y$ and neighbourhoods $V_x^k \in \mathcal{N}_x$ with

$$f(V_x^k \cap K_{\mu k}) \subseteq \overline{W}_{f(x)} \cup N_W^k.$$

Then by introducing a $V_x \in \mathcal{N}_x$ with $V_x \subseteq G_x \cap \bigcap_{k \leqslant n} V_x^k$ and the nowhere dense $N_W = \cup\ [N_W^k : k \leqslant n] \subseteq Y$ one easily gets $f(V_x) \subseteq W_{f(x)} \cup N_W$.

(vi) Follows from (v).

Theorem 4—(i) All restrictions and graph function of a strongly rarely continuous function are again strongly rarely continuous.

(ii) If $f : X \to Y$ is continuous and $g : Y \to Z$ is strongly rarely continuous then $g.f : X \to Z$ is strongly rarely continuous.

PROOF : (i) Notice that the graph function g_f of f satisfy $g_f = (i_X \times f) \circ j_X$ where $j_X(x) = (x, x)$. Thus if f is s. rarely continuous then g_f will also be strongly rarely continuous by Theorem 7i and Theorem 3ii.

(ii) For any $W_{gf(x)} \in \mathcal{N}_{gf(x)}$, one could find a nowhere dense $N_W \subseteq Z$ and an $U_{f(x)} \in \mathcal{N}_{f(x)}$ with $g(U_{f(x)}) \subseteq W_{gf(x)} \cup N_W$ by the Strongly rare-continuity of g. Since there also exists a $G_x \in \mathcal{N}_x$ with $f(G_x) \subseteq U_{f(x)}$, the statement is now clear.

Theorem 4—(i) δ-continuity, almost continuity S & S, θ-continuity, weak-continuity, strongly rare-continuity and rare-continuity of an open function are equivalent.

(ii) A function f is continuous iff f is strongly rarely continuous and inverses of all closed nowhere dense subsets under f are closed.

(iii) A function f is weakly continuous iff f is s. rarely continuous and it is weakly continuous on $\bar{N}_f$.

PROOF : (i) Every open rarely continuous function is weakly continuous[12] and therefore is almost continuous S & S[19] and consequently is δ-continuous[13].

(ii) Necessity is clear and sufficiency follows from the following lemma and (2) of Theorem 1.

Lemma—The following are equivalent for any function f:

(1) f is continuous at $x \in X$.

(2) $\forall W_{f(x)} \in \mathcal{N}_{f(x)}, \exists\, G_x \in \mathcal{N}_x, f^{-1}(\mathrm{cl}\,(f(G_x) - W_{f(x)}))$ is closed.

(3) $\forall W_{f(x)} \in \mathcal{N}_{f(x)}, \exists\, G_x \in \mathcal{N}_x, x \notin \mathrm{cl}\,(G_x - f^{-1}(W_{f(x)}))$.

PROOF : Notice that (2) $\Rightarrow$ (3) by

$$G_x - f^{-1}(W_{f(x)}) \subseteq f^{-1}(\mathrm{cl}\,(f(G_x) - W_{f(x)})) \subseteq f^{-1}(Y - W_{f(x)}).$$

The other implications are straightforward.

(iii) Necessity is obvious. Now let $f : X \to Y$ be strongly rarely continuous. Then the inclusion $f^{-1}(U) \subseteq N_f \cup (f^{-1}(U))_*$ holds for each open $U \subseteq Y$. In fact if a point $x \in f^{-1}(U) - N_f$ satisfy $x \notin (f^{-1}(U))_*$ then there exist a $G_x \in \mathcal{N}_x$ and $W_{f(x)} \in \mathcal{N}_{f(x)}$ such that

$$(f(G_x))^* \subseteq \overline{W}_{f(x)} \subseteq U \cup \partial U, \quad (f(G_x))^* \cap U = \phi,$$

one therefore would get $(f(G_x))^* \subseteq \partial U$ and so $(f(G_x))^* = \phi$ which is contradicting with $x \notin N_f$. Hence if f is strongly rarely continuous then $X - WC_f \subseteq \bar{N}_f$ is obtained by Theorem 2 (iii) after noticing the formula

$$X - WC_f = \cup\,[\mathrm{cl}\,(f^{-1}(U)) - f^{-1}(\bar{U}) : U \subseteq Y \text{ open}]$$

which is true for any function f where WC_f denotes briefly the set of all weak-continuity points of f. Therefore if a strongly rarely continuous function satisfy $\bar{N}_f \subseteq WC_f$ then $X = WC_f$ follows.

Definition 2—The graph of a function $f : X \to Y$ is called strongly α-closed iff for each $(x, y) \notin G(f)$ there exists a couple $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ so that $(f(G_x) \cap {}_yU)^* = \phi$ or equivalently $\alpha(f(G_x)) \cap U_y = \phi$ where as usually $G(f)$ denotes the graph of f.

Remark 3 : Functions with closed graph have strongly α-closed graph and functions with strongly α-closed graph have α-closed graph. In fact if $G(f) \subseteq X \times Y$ is closed then $(x, y) \notin G(f)$ implies the existence of a $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ with $(G_x \times U_y) \cap G(f) = \phi$ or equivalently $f(G_x) \cap U_y = \phi$. The second statement follows easily from the basic inclusion on graphs $(A \times B) \cap G(f) \subseteq A \times (f(A) \cap B)$. In fact if $(x, y) \notin G(f)$ and f has strongly α-closed graph then there exists a $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ with $((G_x \times U_y) \cap G(f))^* = \phi$ and so $(x, y) \notin (G(f))^* \cup G(f) = \mathrm{cl}_\alpha\, G(f)$ and therefore $\mathrm{cl}_\alpha\, G(f) = G(f)$ are obtained. The function mentioned in Remark 5 in the sequel has non closed but strongly α-closed graph. The function $f : E^1 \to E^1$ defined as $f(x) = x - \sup((-\infty, x] \cap Z)$ has α-closed but not strongly α-closed graph since $G(f) \subseteq E^1 \times E^1$ satisfy $(G(f))^* = \phi$ but $f(G_k) \cap U_1)^* \neq \phi$ holds for each $(k, 1) \notin G(f)$, $k \in Z$ where Z denotes as usually the set of all integers.

Remark 4 : It is known that a function with the closed graph defined into a H-closed space is not necessarily weakly continuous[6] but is rarely continuous.[12]

Theorem 5—(i) If $f : X \to Y$ has strongly α-closed graph then $(f^{-1}(K))_* \subseteq f^{-1}(K)$ holds for any almost compact $K \subseteq Y$.

(ii) Functions with strongly α-closed graph defined into almost compact spaces are strongly rarely continuous.

(iii) Functions with closed graph defined into H-closed spaces are strongly rarely continuous.

(iv) The following are equivalent for functions defined into H-closed spaces:

(1) f is strongly rarely continuous.

(2) The graph of f is strongly α-closed.

(3) $(f^{-1}(K))_* \subseteq f^{-1}(K)$ hold for all almost compact $K \subseteq Y$.

PROOF : (i) Let $x \notin f^{-1}(K)$. Then there exists a couple $(G_x(y), U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ with $(f(G_x(y)) \cap U_y)^* = \phi$ and therefore $(f(G_x(y)) \cap \bar{U}_y)^* = \phi$ for each $y \in K$. Since K is an almost compact subspace, there exists a $G_x \in \mathcal{N}_x$ with $(f(G_x) \cap K)^* = \phi$. Hence $x \notin (f^{-1}(K))_*$ is established.

(ii) Take any open $U \subseteq Y$. Then $(f^{-1}(U))_* = (f^{-1}(\bar{U}))_* \subseteq f^{-1}(\bar{U})$ since all regularly closed subsets in an almost compact space are almost compact subspaces. The item now follows from Theorem 2(iii).

(iii) Direct consequence of (ii).

(iv) The implication (1) $\Rightarrow$ (2) will be proved in Theorem 6i. The others were already established.

Corollary (Long and Herrington[12])—Let $f : X \to Y$ be a function with closed graph where Y is H-closed. Then f is rarely continuous.

Theorem 6—(i) The graph of a strongly rarely continuous function defined into a T_2 space is strongly α-closed but not necessarily closed.

(ii) Images of compact subspaces under a function with strongly α-closed graph are α-closed.

(iii) Images of compact subspaces under a strongly rarely continuous function defined into a T_2 spaces are α-closed but not necessarily closed.

PROOF : (i) If $(x, y) \notin G(f)$ then there exists a couple $(W_y, G_x) \in \mathcal{N}_y \times \mathcal{N}_x$ with $\alpha f(G_x) \cap \bar{W}_y = \phi$ since f is strongly rarely continuous and Y is a T_2 space. Hence $\alpha(f(G_x) \cap W_y) = \phi$ and consequently $(f(G_x) \cap W_y)^* = \phi$ are found.

(ii) Let $K \subset X$ be compact and $y \notin f(K)$. Then there exists a $(G_x, U_y(x)) \in \mathcal{N}_x \times \mathcal{N}_y$ with $(f(G_x) \cap U_y(x))^* = \phi$ or equivalently $(f(G_x))^* \cap U_y(x) = \phi$ for each $x \in K$ since $G(f)$ is strongly α-closed. Then it is easy to see that there exists a $U_y \in \mathcal{N}_y$ with $(f(K))^* \cap U_y = \phi$ i. e. $(f(K) \cap U_y)^* = \phi$. Hence $y \notin (f(K))^*$ and so $y \notin \mathrm{cl}_\alpha f(K)$ arc found.

(iii) Follows from the first two items.

Remark 5 : The classical step function proves that a strongly rarely continuous function does not necessarily have closed graph even it is defined between two metriz-

able spaces since $(k+1, k) \in \overline{G(f)} - G(f)$ holds for each integer k in this example. Notice also that the function f defined in Remark 1 from E^1 into the cofinite topology on reals is strongly rarely continuous (even δ-continuous) but the images of compact subspaces are not necessarily closed.

Remark 6 : One way for obtaining the new strongly rarely continuous functions from the old ones is to product them by the first item of the following theorem just as for continuous and weakly type of continuous functions another way is to take their compositions with continuous functions. Another way is to take their compositions with continuous functions just as in Theorem (3ii).

Theorem 7—(i) A product function is s. rarely continuous iff each factor function is so.

(ii) A function f defined into a product space is strongly rarely continuous if each composition of f with the projection mappings is strongly rarely continuous.

PROOF : (i) Let each $f_\nu : X_\nu \to Y_\nu$ be strongly rarely continuous and the point $x = (x_\nu)_\nu \in \Pi X_\nu$ is taken. Then the closure of any basic neighbourhood of $(y_\nu)_\nu = (\Pi f_\nu)(x) = (f_\nu(x_\nu))$ contains $\cap [\pi^{-1}_{\nu_k}(\overline{W}(y_{\nu_k}) : k \leqslant n]$ i.e. the closure of a base member of the product space where $W(y_{\nu_k}) \in \mathcal{N}(y_{\nu_k})$ in the space Y_{ν_k} and π_ν denotes the projection mapping onto X_ν. Therefore there exists a $G(x_{\nu_k}) \in \mathcal{N}(x_{\nu_k})$ with $\alpha\ f_{\nu_k}(G(x_{\nu_k})) \subseteq \overline{W}(y_\nu)$ for each $k \leqslant n$. Then one gets

$$\alpha(f_0(\bigcap_{k \leqslant n} \pi^{-1}_{\nu_k}(G(x_{\nu_k}))) \subseteq \alpha(\bigcap_{k \leqslant n} \pi^{-1}_{\nu_k}(f_{\nu_k}(G(x_{\nu_k})))$$

$$= \bigcap_{k \leqslant n} \pi^{-1}_{\nu_k}(\alpha\ f_{\nu_k}(G(x_{\nu_k}))$$

where f_0 denotes briefly the product function πf_ν. Hence the product function f_0 is strongly rarely continuous at $x = (x_\nu)_\nu$. Now conversely let f_0 be strongly rarely continuous. Take any index γ and any point $x_\gamma \in X_\gamma$. Then one easily creates a point x in the product space with $\pi_\gamma(x) = x_\gamma$. So for any $W(f_\gamma(x_\gamma)) \in \mathcal{N}(f_\gamma(x_\gamma))$ one gets

$$\pi^{-1}_\gamma(\overline{W}(f_\gamma(x_\gamma))) \supseteq \alpha(f_0(\bigcap_{k \leqslant n} \pi^{-1}_{\nu_k}(G(x_{\nu_k})))$$

$$= \bigcap_{k \leqslant n} \pi^{-1}_{\nu_k}(\alpha f_{\gamma_k}(G(x_{\gamma_k}))) \cap \prod_\nu \alpha\ f_\gamma(X_\nu)$$

by the appropriately chosen neighbourhoods $G(x_{\nu_k}) \in \mathcal{N}(x_{\nu_k})$. Now whether the index γ satisfy $\gamma = \nu k$ for a $k \leqslant n$ or not, the function f_γ will be strongly rarely continuous at $x_\gamma \in X_\gamma$ by taking the π_γ projections of the both sides.

(ii) Left to the reader.

Remark 7 : The first item of the following theorem is a slight generalization of Theorem 6 of Rose[17]. The second item, stated independently by Takashi Noiri and the author gives a different version of a result of Noiri which is expressed in one of our correspondencies : Rarely continuous semi-open functions[2] are weakly continuous. It is also a strength version of Theorem 4i.

Theorem 8—(i) $f : X \to Y$ is almost continuous H. iff $f(\bar{U}) \subseteq \mathrm{cl}\, f(U)$ for all semi-open $U \subseteq X$.

(ii) Almost continuity S & S, weak-continuity and strongly rare-continuity of an almost open Rose function are equivalent.

PROOF : (i) We give here an independent proof. Let $f : X \to Y$ be almost continuous H. and $U \subseteq X$ be semi-open. Take any $x \in \bar{U} = \beta\, U$. Then $W_{f(x)} \cap f(U) = f(U \cap f^{-1}(W_{f(x)}))$ is nonempty since

$$\mathrm{cl}\,(U \cap f^{-1}(W_{f(x)})) \supseteq \mathrm{cl}\,(\overset{\circ}{U} \cap \mathrm{cl}\, f^{-1}(W_{f(x)})) \supseteq \beta\, U \cap \alpha\,(f^{-1}(W_{f(x)}) \neq \phi$$

are all nonempty for each $W_{f(x)} \in \mathcal{W}_{f(x)}$. Sufficiency is a consequence of Theorem 6 of Rose[17].

(ii) A function $f : X \to Y$ is called almost open Rose [17] iff $f(G) \subseteq \alpha\, f(G)$ holds for each open $G \subseteq X$, i. e. images of all open sets are preopen. So almost continuity S and S of almost open Rose, s. rarely continuous functions follows easily by Theorem ii.

WEAK*CONTINUITY

We define the weak*-continuous function as the strongly rarely continuous function f with the nowhere dense or equivalently empty N_f set. Then it is easy to see that by Theorem 4 (iii) every weak*-continuous function is weakly continuous but not conversely. Notice that a strongly rarely continuous function is not necessarily weak*-continuous. This is immediate after considering the step functions. Therefore the following implications are not reversible

weak*-continuity $\Rightarrow$ weak-continuity $\Rightarrow$ strongly rare-continutity.

If f is weak*-continuous then $(f^{-1}(U))^* \subseteq (f^{-1}(U))_*$ holds for each semi-open $U \subseteq Y$. Hence by remembering the equivalency of $G \cap A_* \neq \phi$ iff $(G \cap A^* \neq \phi$ proved in preparations, we obtain the following characterization :

Lemma—$f : X \to Y$ is weak*-continuous iff $(G \cap f^{-1}(U))^* \neq \phi$ imply $(f(G) \cap U)^* \neq \phi$ for each open G and for each semi-open $U \subseteq Y$.

Hence the first item of the following theorem which is version of Theorem 8 (i) is derived.

Theorem 9—(i) A weak*-continuous $f : X \to Y$ is almost continuous H. iff $f(\bar{U}) \subseteq (f(U))^*$ for each semi-open $U \subseteq X$.

(ii) Injective weak*-continuous functions into T_2 spaces can only be defined on T_2 spaces.

(iii) Strongly rare-continuity and weak*-continuity are equivalent for almost open Rose functions.

(iv) Strongly rare-continuity and weak*-continuity are equivalent for almost open S & S almost continuous H. (resp. semi-continuous) functions.

PROOF : (i) Notice only that if almost continuous H. function f is weak*-continuous then the following will be obtained by using the same notation of the proof of Theorem 8 (i)

$$(\text{cl}\,(U \cap f^{-1}\,(W_{f(x)})))^* \supseteq (\text{cl}\,(\overset{\circ}{U} \cap \alpha f^{-1}\,(W_{f(x)})))^*$$
$$\supseteq \text{cl}\,(\beta\, U \cap \alpha f^{-1}\,(W_{f(x)})).$$

(ii) Let $f : X \to X$ be injective and weak*-continuous and let Y be a T_2 space. Then if $x \neq \xi$ there exists a couple $(G_x, W_{f(\xi)}) \in \mathscr{N}_x \times \mathscr{N}_{f(\xi)}$ with $\alpha f(G_x) \subset \overline{W}_{f(\xi)}$ $\cap = \phi$ and so there exists a $U_\xi \in \mathscr{N}_\xi$ with $\alpha f(G_x) \cap \alpha f(U_\xi) = \phi$. Hence $\alpha f(G_x\ U_\xi) = \phi$ and consequently $G_x \cap U_\xi = \phi$ are found by the hypothesis of $N_f = \phi$. So different points have disjoint basic neighbourhoods in χ.

(iii) Notice only that $N_f = \phi$ holds for any almost open Rose function f since $f(G_x)$ is contained in $\alpha f(G_x)$ for any $G_x \in \mathscr{N}_x$.

(iv) These statements are straightforward consequences of Theorem 4.2 and Theorem 4.3 of Noiri's paper[14] after (iii).

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QUASI-STATIC RESPONSE OF A LAYERED HALF-SPACE TO SURFACE LOADS

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The closed form expressions for the stresses caused by a two-dimensional shear line load acting on the boundary of a semi-infinite medium consisting of a homogeneous elastic layer lying over a homogeneous elastic half-space are first derived. The correspondence principle of viscoelasticity is then used to obtain the quasi-static response when the elastic half-space is replaced by a Maxwell viscoelastic half-space. Numerical calculations performed indicate that the quasi-static stresses differ significantly from the corresponding static stresses when the medium is purely elastic.

INTRODUCTION

The well known Boussinesq solution to the problem of a normal static load on the surface of a semi-infinite elastic medium offers wide applications to loading problems in geophysics and engineering. Transient crustal movement due to surface loading is of considerable interest in understanding the rheology of the earth's crust and upper mantle. This phenomenon is considered to be controlled by a quasi-static process of stress relaxation is viscoelastic regions within the earth. In quasi-static processes, the stress equilibrium exists at every point at each instant of time. This permits the neglect of the inertia term in the equation of motion. The quasi-static behaviour of the system is thus determined by the equation of equilibrium and equations relating stress, strain and displacement subject to boundary or initial conditions. The quasi-static deformation of a viscoelastic half-space by surface loads has been discussed by Lee[1], Fung[2]. Christensen[3], and others.

In the present paper, we obtain the static stresses due to a shear line load acting at the boundary of a semi-infinite medium which consists of a homogeneous, isotropic, elastic layer lying over a homogeneous, isotropic, elastic half-space. The correspondence principle of linear viscoelasticity is then used to obtain the quasi-static stresses when the half-space is Maxwell viscoelastic. In this model, the elastic layer represents the lithosphere of the earth and the Maxwell viscoelastic half-space represents

the asthenosphere. Graphs for the shear stresses are drawn. It is found that these graphs differ significantly from the corresponding graphs for the elastic case.

2. Formulation of the Problem

We consider a model consisting of a homogeneous, isotropic, elastic layer of thickness H lying over a homogeneous, isotropic, Maxwell viscoelastic half-space. We place the origin of a cartesian coordinate system (x, y, z) at the boundary of the semi-infinite medium and the z-axis is drawn into the medium (Fig. 1). Let a shear line

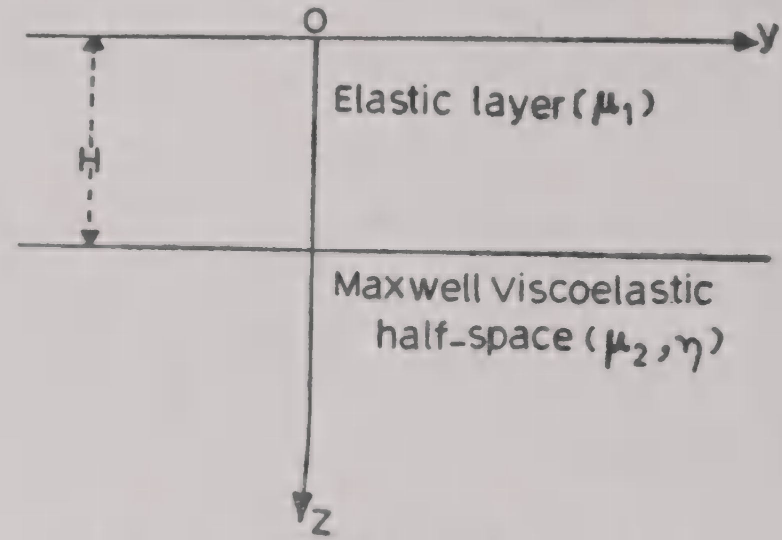

Fig. 1. Section of the model by the plane $x = 0$.

load R per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the x-axis. We shall be considering an antiplane strain problem in which the displacement components are given by

$$u = u(y, z), \; v = w \equiv 0.$$

We first calculate the shear stresses p_{12} and p_{13} at any point of the medium caused by a shear line load R acting on the surface $z = 0$ of the corresponding elastic model. We then use the corresponding correspondence principle of linear viscoelasticity to obtain the quasi-static response.

3. Elastostatic Solution

In the antiplane strain problem, the displacement n (for zero body force) satisfies the equation

$$\nabla^2 u = 0 \qquad \ldots(3.1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad \ldots(3.2)$$

A suitable solution of (3.1) is of the type

$$u = \int_0^\infty \left(Ae^{-kz} + Be^{kz} \right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk \qquad \text{...(3.3)}$$

where A, B may be functions of k. Then

$$p_{12} = \mu \frac{\partial u}{\partial y} = \mu \int_0^\infty (Ae^{-kz} + Be^{kz}) \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k\, dk \qquad \text{...(3.4)}$$

$$p_{13} = \mu \frac{\partial u}{\partial z} = \mu \int_0^\infty (-Ae^{-kz} + Be^{kz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \qquad \text{...(3.5)}$$

where μ is the rigidity of the medium.

We consider a semi-infinite medium consisting of a homogeneous, isotropic, elastic layer of thickness H lying over a homogeneous, isotropic, elastic half-space. It is assumed that the layer and the half-space are in welded contact. Let a shear line load R per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the x-axis. Then the boundary condition at $z = 0$ is

$$p_{13} = -R\,\delta(y) \qquad \text{...(3.6)}$$

where $\delta(y)$ denote the Dirac delta fudction. We use the representation

$$\delta(y) = \int_0^\infty \cos ky\, dk. \qquad \text{...(3.7)}$$

Equations (3 6) and (3.7) suggest that we must choose the lower solution ($\cos ky$) in the expression (3.3) for u and other results related to u. The displacement u and the shear stress p_{13} at any point are

$$u(z) = \begin{cases} \int_0^\infty (A_1 e^{-kz} + B_1 e^{kz}) \cos ky\, dk & 0 \leqslant z \leqslant H \\ \int_0^\infty A_2 e^{-kz} \cos ky\, dk & z \geqslant H \end{cases} \qquad \text{...(3.8)}$$

$$p_{13}(z) = \begin{cases} \mu_1 \int_0^\infty (-A_1 e^{-kz} + B_1 e^{kz}) \cos ky\, k\, dk & 0 \leqslant z \leqslant H \\ -\mu_2 \int_0^\infty A_2 e^{-kz} \cos ky\, k\, dk & z \geqslant H \end{cases} \qquad \text{...(3.9)}$$

where A_1, B_1 and A_2 are to be determined with the help of boundary conditions. μ_1 is the rigidity of the isotropic elastic layer and μ_2 is the rigidity of the isotropic elastic half-space. In the solution for the region $z \geqslant H$, the coefficient of $\exp(kz)$ is taken as zero, since, otherwise, $u \to \infty$ as $z \to \infty$. Using the boundary condition (3.6) and the continuity of the displacement (u) and the shear stress (p_{13}) across $z = H$, we can determine the coefficients A_1, B_1 and A_2. The substitution of the values of these coefficients in (3.8) gives the displacement u at any point of the medium. The corresponding stresses can then be obtained by simple differentiation. Expanding the denominator in a power series and evaluating the integrals, we find

$$p_{12} = \frac{-1}{\pi}\left[N_0\left(\frac{y}{y^2+z^2}\right) + \sum_{n=1}^{\infty} N_n\left\{\frac{y}{y^2+(2nH-z)^2} + \frac{y}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.10)$$

$$p_{13} = \frac{-1}{\pi}\left[N_0\left(\frac{z}{y^2+z^2}\right) - \sum_{n=1}^{\infty} N_n\left\{\frac{2nH-z}{y^2+(2nH-z)^2} - \frac{2nH+z}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.11)$$

for $0 \leqslant z < H$ and, for $z \geqslant H$

$$p_{12} = \frac{-2}{\pi}\left[M_0\left(\frac{y}{y^2+z^2}\right) + \sum_{n=1}^{\infty} M_n\left\{\frac{y}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.12)$$

$$p_{13} = \frac{-2}{\pi}\left[M_0\left(\frac{z}{y^2+z^2}\right) + \sum_{n=1}^{\infty} M_n\left\{\frac{2nH+z}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.13)$$

where

$$N_0 = R,\; M_0 = R\left(\frac{\mu^2}{\mu_1+\mu_2}\right) \quad \ldots(3.14)$$

$$N_n = R\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^n,\; M_n = R\left[\frac{\mu_2(\mu_1-\mu_2)^n}{(\mu_1+\mu_2)^{n\cdot 1}}\right]. \quad (3.15)$$

4. Viscoelastic Solution

We now use the correspondence principle[4] to obtain the quasi-static shear stresses for a model consisting of an elastic layer lying over a Maxwell viscoelastic half-space. For the elastic layer

$$p_{12} = 2\mu_1 e_{12},\; p_{13} = 2\mu_1 e_{13}. \quad \ldots(4.1)$$

For the Maxwell viscoelastic half-space

$$e_{12} = \frac{1}{2\mu_2} p_{12} + \frac{1}{\eta} p_{12} \qquad \ldots(4.2a)$$

$$e_{13} = \frac{1}{2\mu_2} p_{13} + \frac{1}{\eta} p_{13} \qquad \ldots(4.2b)$$

where η is viscosity and the dot ($\cdot$) signifies time-differentiation. Taking the Laplace transform of (4.2a, b) we obtain

$$s\,\bar{e}_{12} = \frac{s}{2\mu_2}\,\bar{p}_{12} + \frac{1}{\eta}\,\bar{p}_{12} \qquad \ldots(4.3a)$$

$$s\,\bar{e}_{13} = \frac{s}{2\mu_2}\,\bar{p}_{13} + \frac{1}{\eta}\,\bar{p}_{13} \qquad \ldots(4.3b)$$

where s is the Laplace transform variable. We may write (4.3a,b) in the form

$$\bar{p}_{12} = 2\mu_2^*\,\bar{e}_{12},\ \bar{p}_{13} = 2\mu_2^*\,\bar{e}_{13} \qquad \ldots(4.4)$$

where

$$\mu_2^* = \frac{s\mu_2}{s + 2\,\tau^{-1}} \qquad \ldots(4.5)$$

is the transform rigidity and $\tau = \eta/\mu_2$ is the relaxation time. Time-dependence of the load function is taken to be a step-function i. e.,

$$R\,(t) = R_0\,H\,(t) \qquad \ldots(4.6)$$

where $H\,(t)$ is the Heaviside step function. Then

$$\overline{R\,(t)} = \frac{R_0}{s}. \qquad \ldots(4.7)$$

In order to find the Laplace transformed solution of the viscoelastic problem, it is only necessary to replace μ_2 and R by μ_2^* and $\bar{R}$, respectively, in the corresponding elastic solution. From (3.10) – (3.13), we notice that μ_2 and R occur only in the expressions for N_0, N_n, M_0 and M_n. Therefore, the Laplace transformed solution of the viscoelastic problem is obtained from (3.10)—(3.13) on replacing N_0, N_n, M_0 and M_n by $\bar{N}_0$, $\bar{N}_n$, $\bar{M}_0$ and $\bar{M}_n$, respectively, where from (3.14), (3.15), (4.5) and (4.7).

$$\bar{N}_0 = \frac{R_0}{s},\ \bar{M}_0 = \frac{R_0}{C}\left(\frac{1}{s + A}\right) \qquad \ldots(4.8)$$

$$\bar{N}_n = R_0\,S_n\,(s),\ \bar{M}_n = \frac{R_0}{C}\,G_n\,(s) \qquad \ldots(4.9)$$

$$S_n\,(s) = \frac{(s\,B + A)^n}{s\,(s + A)^n},\ G_n\,(s) = \frac{(sB + A)^n}{(s + A)^{n+1}} \qquad \ldots(4.10)$$

$$A = \frac{2\mu_1}{(\mu_1 + \mu^2)\,\tau},\ B = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2},\ C = \frac{\mu_1 + \mu_2}{\mu^2}. \qquad ..(4.11)$$

In order to find the inverse Laplace transforms of $S_n(s)$ and $G_n(s)$, we use transform integrals listed in Erdélyi[5]. We find

$$L^{-1}[S_n(s)] = 1 + \exp(-At)\sum_{m=1}^{n} \frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m} \qquad ...(4.12)$$

$$L^{-4}[G_n(s)] = \left[\frac{(1-B)^n (At)^n}{n!} + \sum_{m=1}^{n} \frac{B^m (1-B)^{n-m}}{(n-m)!\,m!}(A\,t)^{n-m}\right]$$
$$\exp(-At) \qquad ...(4.13)$$

$$L^{-1}\left[\frac{1}{+A}\right] = \exp(-At) \qquad ..(4.14)$$

where

$$F_{2m}(s) = \frac{d^{m-1}}{ds^{m\ 1}}\left[\frac{(sB+A)^n}{s}\right]. \qquad ...(4.15)$$

Equations (3.10) – (3.13), (4.8), (4.9), (2.12) – (4.14) yield ($t > 0$)

$$p_{12} = \frac{-R_0}{\pi}\left[\frac{y}{y^2+z^2} + \sum_{n=1}^{\infty}\left\{1 + \exp(-At)\sum_{m=1}^{n}\right.\right.$$
$$\left.\frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m}\right\} \times \left\{\frac{y}{y^2+(2nH+z)^2}\right.$$
$$\left.\left. + \frac{y}{y^2+(2nH+z)^2}\right\}\right] \qquad ...(4.16)$$

$$p_{13} = \frac{-R_0}{\pi}\left[\frac{z}{y^2+z^2} - \sum_{n=1}^{\infty}\left\{1 + \exp(-At)\sum_{m=1}^{n}\right.\right.$$
$$\left.\times \frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m}\right\} \times \left\{\frac{2n\,H - z}{y^2+(2nH-z)^2}\right.$$
$$\left.\left. - \frac{2n\,H + z}{y^2+(2nH+z)^2}\right\}\right] \qquad ...(4.17)$$

for $0 \leqslant z \leqslant H$ and, for $z \geqslant H$

$$p_{12} = \frac{-2R_0}{\pi C}\left[\frac{y}{y^2+z^2} + \sum_{n=1}^{\infty}\left\{\frac{y}{y^2+(2nH+z)^2}\right\}\left\{\frac{(1-B)^n\,(A\,t)^n}{n!}\right.\right.$$

(equation continued on p. 627)

$$+ \sum_{m=1}^{n} \frac{B^m (1 - B)^{n-m}}{(n - m)!\, m!} (At)^{n-m} \Big\}\Big] \exp(-At) \qquad \ldots(4.18)$$

$$p_{13} = \frac{-2R_0}{\pi C} \left[\frac{z}{y^2 + z^2} + \sum_{n=1}^{\infty} \left\{ \frac{2nH + z}{y^2 + (2nH + z)^2} \right\} \right.$$

$$\times \left\{ \frac{(1 - B)^n (At)^n}{n!} + \sum_{m=1}^{n} \frac{B^m (1 - B)^{n-m}}{(n - m)!\, m!} (At)^{n-m} \right\} \Big]$$

$$\times \exp(At). \qquad \ldots(4.19)$$

Equation (4.16) and (4.17) give the quasi-static shear stresses at any point of an elastic layer of thickness H lying over a Maxwell viscoelastic half-space caused by a shear line load acting at the origin to the surface $z = 0$ in the positive direction of the x-axis. Equations (4.18) and (4.19) give the quasi-static shear stresses at any point of the Maxwell viscoelastic half-space.

5. Particular Case

We consider the particular case when the rigidities μ_1 and μ_2 are equal, i. e.,

$$\mu_1 = \mu_2 = \mu \text{ (say)}. \qquad \ldots(5.1)$$

Equations (4.11) and (5.1) give

$$A = \tau^{-1}, \; B = 0, \; C = 2. \qquad \ldots(5.2)$$

Using (5.2) equation (4.15) yields

$$F_{2m}(-A) = \frac{-(m-1)!}{\tau^{n-m}} . \qquad \ldots(5.3)$$

From (4.16) – (4.19), (5.2) and (5.3), we find ($t > 0$)

$$p_{12} = \frac{-R_0}{\pi} \left[\frac{y}{y^2 + z^2} + \sum_{n=1}^{\infty} \left\{ 1 - \exp(-t/\tau) \sum_{k=1}^{n-1} \frac{(t/\tau)^k}{k!} \right\} \right.$$

$$\times \left\{ \frac{y}{y^2 + (2nH - z)^2} + \frac{y}{y^2 + (2nH + z)^2} \right\} \Big] \qquad \ldots(5.4)$$

$$p_{13} = \frac{-R_0}{\pi} \left[\frac{z}{y^2 + z^2} - \sum_{n=0}^{\infty} \left\{ 1 - \exp(-t/\tau) \sum_{k=0}^{n-1} \frac{(t/\tau)^k}{k!} \right\} \right.$$

$$\times \left\{ \frac{2nH - z}{y^2 + (2nH - z)^2} - \frac{2nH + z}{y^2 + (2nH + z)^2} \right\} \Big] \qquad \ldots(5.5)$$

for $0 \leqslant z < H$ and

$$p_{12} = \frac{-R_0}{\pi}\left[\frac{y}{y^2+z^2} + \sum_{n=1}^{\infty}\frac{(t/\tau)^n}{n!}\left\{\frac{y}{y^2+(2nH+z)^2}\right\}\right]$$

$$\exp(-t/\tau) \qquad \ldots(5.6)$$

$$p_{13} = \frac{-R_0}{\pi}\left[\frac{z}{y^2+z^2} \sum_{n=1}^{\infty}\frac{(t/\tau)^n}{n!}\left\{\frac{2nH+z}{y^2+(2nH+z)^2}\right\}\right]$$

$$\exp(-t/\tau) \qquad \ldots(5.7)$$

for $z \geqslant H$.

The case when $t = 0$ corresponds to the elastic problem. Equations (3.10)—(3.13) and (5.1) then yield

$$p_{12} = \frac{-R_0}{\pi}\left(\frac{y}{y^2+z^2}\right) \qquad \ldots(5.8a)$$

$$p_{13} = \frac{-R_0}{\pi}\left(\frac{z}{y^2+z^2}\right) \qquad \ldots(5.8b)$$

for every value of z $(0 \leqslant z < \infty)$ with $R = R_0$.

Since we have taken $\mu_1 = \mu_2 = \mu$, eqns. (5.8a, b), in fact, give the shear stresses at any point z within an elastic half-space due to a shear line load R per unit length acting at the origin to the surface $z = 0$ in the positive direction of the x-axis.

6. Numerical Results

In eqns. (5.4) and (5.5), we have obtained the expressions for the quasi-static shear stresses p_{12} and p_{13} within the medium $(0 \leqslant z \leqslant H)$ caused by a shear line load acting at the origin to the surface $z = 0$ in the positive direction of the x-axis. We wish to study the variation of these stresses with horizontal distance y and time t for given values of z. For this purpose, we define the dimensionless quantities α, Y, T, P_{12} and P_{13} through the relations

$$z = \alpha H,\ y = YH,\ t = T\tau \qquad \ldots(6.1)$$

$$p_{12} = \frac{R_0}{\pi H} P_{12},\ p_{13} = \frac{R_0}{\pi H} P_{13}.$$

Using (6.1), equations (5.4) and (5.5) yield $(T > 0)$

$$P_{12} = -\frac{Y}{Y^2+\alpha^2} - \sum_{n=1}^{\infty}\left\{1 - \exp(-T)\sum_{k=0}^{n-1}\frac{T^k}{k!}\right\}\left\{\frac{Y}{Y^2+(2n-\alpha)^2}\right.$$

(equation continued on p. 629)

$$+ \frac{Y}{Y^2 + (2n + \alpha)^2} \Bigg\} \qquad \ldots(6.2)$$

$$p_{13} = - \frac{\alpha}{Y^2 + \alpha^2} + \sum_{n=1}^{\infty} \left\{ 1 - \exp(-T) \sum_{k=0}^{n-1} \frac{T^k}{k!} \right\} \left\{ \frac{2n - \alpha}{Y^2 + (2n - \alpha)^2} \right.$$

$$\left. - \frac{2n + \alpha}{Y^2 + (2n + \alpha)^2} \right\} \qquad \ldots(6.3)$$

where P_{12} and P_{13} are the dimensionless shear stresses. Y and T are, respectively, the dimensionless horizontal distance and the dimensionless time. For $T = 0$ eqns. (5.8a, b) give

$$P_{12} = - \frac{Y}{Y^2 + \alpha^2} \qquad \ldots(6.4)$$

$$P_{13} = - \frac{\alpha}{Y^2 + \alpha^2} \qquad \ldots(6.5)$$

as the dimensionless stresses for the elastic case. We note that

$$\frac{Y}{Y^2 + (2n - \alpha)} + \frac{Y}{Y^2 + (2n + \alpha)^2}$$

$$= \frac{2Y(4n^2 + Y^2 + \alpha^2)}{[Y^2 + (2n - \alpha)^2][Y^2 + (2n + \alpha)^2]} \qquad \ldots(6.6)$$

and

$$\frac{2n - \alpha}{Y^2 + (2n - \alpha)^2} - \frac{2n + \alpha}{Y^2 + (2n + \alpha)^2}$$

$$= \frac{2\alpha[4n^2 - (Y^2 + \alpha^2)]}{[Y^2 + (2n - \alpha)^2][Y^2 + (2n + \alpha)^2]} \cdot \qquad \ldots(6.7)$$

Since $\left\{ \exp(-T) \sum_{k=0}^{n-1} \frac{T^k}{k} \right\} < 1$ for all values of n and $T > 0$, it is obvious that the infinite series appearing on the right hand sides of (6.2) and (6.3) converge at least as rapidly as the infinite series $\Sigma n \frac{1}{2}$. In our numerical computation, we found that the first ten terms of this infinite series are adequate.

Figures 2-4 exhibit the variation of the dimensionless horizontal shear stress P_{12} with the dimensionless horizontal distance Y for three values of z, namely, $z = H$, $H/2$, 0 and three values of the dimensionless time T ($T = 0, 1, 10$). For all values of T and $z > 0$, $P_{12} = 0$ when $Y = 0$ and when $z = 0$, P_{12} tends to infinity as Y tends to zero [see eqns. (6.2) and (6.4)]. The graphs for $T = 0$ correspond to the elastic case. From (6.4), we find that, in the elastic case, the shear stress P_{12}, for all values

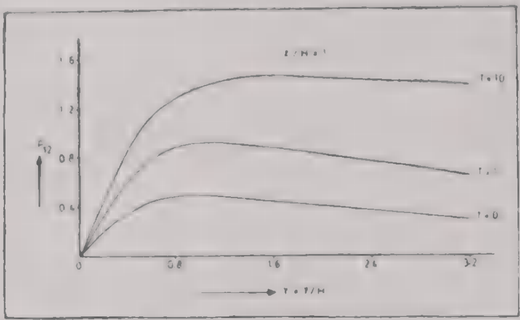

FIG. 2. Variation of the shear stress P_{12} with the horizontal distance Y when $z = H$.

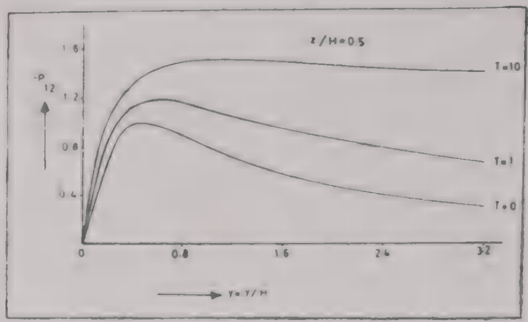

FIG. 3. Variation of the shear stress P_{12} with the horizontal distance Y when $z = H/2$.

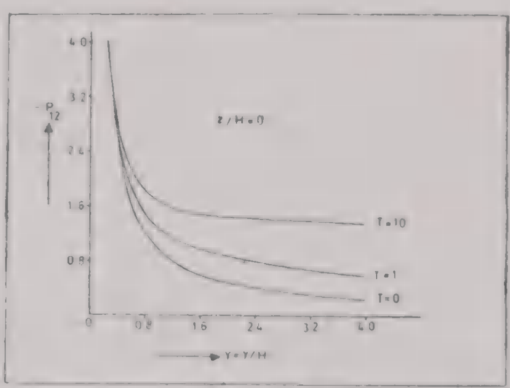

FIG. 4. Variation of the shear stress P_{12} with the horizontal distance Y when $z = 0$.

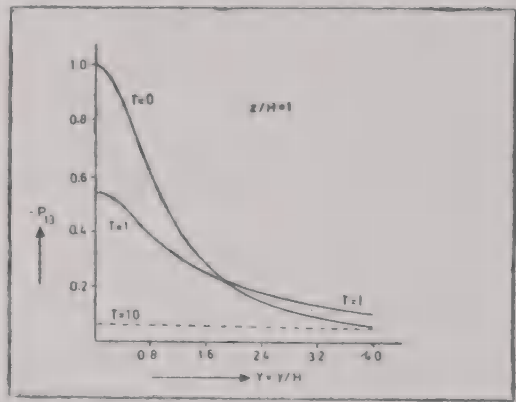

FIG. 5. Variation of the shear stress P_{13} with the horizontal distance Y when $z = H$.

of z, tends to zero as Y tends to infinity. We note that the deviation of the viscoelastic solution from the elastic solution increases as z increases for a given value of Y.

Figures 5-6 show the variation of the dimensionless shear stress P_{13} with the dimensionless horizontal distance Y for two values of z, namely, $z = H, H/2$ and different values of the dimensionless time T. The graphs for $T = 0$ correspond to the

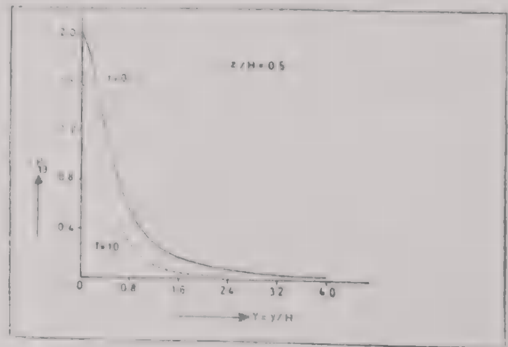

FIG. 6. Variation of the shear stress P_{13} with the horizontal distance Y when $z = H/2$.

elastic case for which $P_{13} = -H/z$ when $Y = 0$. Also, P_{13}, for all values of z, tends to zero as Y approaches infinity [see eqn. (6.5)]. For $z = H$, the graphs for various values of T are quite different from the graph for the elastic case (Fig. 5). For $z = H/2$, there is only a slight difference between the graphs for the viscoelastic case and the elastic case (Fig. 6).

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A NOTE ON THE SQUEEZE FILM LUBRICATION WITH NON-NEWTONIAN FLUID

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A systematic analytical study of squeeze film lubrication between two approaching parallel surfaces is presented by considering second order fluid as lubricant. The closed form expressions for the average pressure distribution, load carrying capacity and time of approach have been obtained. The pressure and load capacity are found to increase significantly as compared to the Newtonian case. The time-height relation indicates that the time of approach for second order fluid is delayed considerably in comparison with the Newtonian fluid of the same viscosity.

Nomenclature

A_1, A_2	=	The first two Rivlin-Ericksen tensors
f_1, f_2	=	functions of h [defined eqn. (23)]
$h(t)$	=	film thickness at time t
$h(t_0)$	=	film thickness at time $t = t_0$ a reference time
$H = \dfrac{h(t)}{h(t_0)}$	=	Ratio of film thickness at any two times t and t_0
L	=	Characteristic length
m, n	=	Functions of x given by eqns. (11 & 12)
N_n	=	Dimensionless parameter [defined in eqn. (27)]
p	=	Pressure in the film region
P	=	Average pressure in the film region
$\bar{P}$	=	Dimensionless average pressure [defined in (eqn. 24)]
S	=	Stress tensor
T	=	Time of approach
$\bar{T}$	=	Dimensionless time of approach [defined in eqn. (26)]

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u, v	=	Velocity components in x & y directions respectively
V_0	=	Velocity of approach
V	=	Velocity vector
W	=	Load carrying capacity
$\overline{W}$	=	Dimensionless load carrying capacity (defined in eqn. 25)
x, y	=	Cartesian co-ordinates
$\bar{x}, \bar{y}$	=	Dimennsionless cartesian co-ordinates
$\epsilon = \frac{h}{L}$	=	Dimensionless film thickness
$\mu = \phi_0$	=	Newtonian viscosity
ρ	=	Density
ϕ_i	=	Material constants of the fluid ($i = 0, 1, 2$).

1. Introduction

The mechanism of lubrication of machine parts, that come into direct contact is aimed at the elimination of destructive heating, minimisation of wear and increasing mechanical efficiency. The lubricant prevents direct contact of surfaces with one another and forms uninterrupted fluid film between two mutually opposing parts of machine to which it is applied. When two surfaces containing lubricant in between them approach each other, then the fluid is squeezed resulting in the build up of pressure which helps in avoiding the possible contact of surfaces. This is termed as squeeze film lubrication. The relevant literature on squeeze film lubrication can be found in Moore[1] and Archibald[2]. The lubricative action depends mainly on the material properties of the lubricating fluid. An effective lubricant should possess a preferable degree of viscosity and should be chemically stable and inert towards metals. Most of the commonly used lubricants are thick polymer solutions exhibiting rheological characreristics such as normal stress differences in shear flow. The rheological behaviour of thick polymer solutions can be adequatety described by the constitutive equation for the second order fluids due to Coleman and Noll[3]. The objective of the present article is to study the squeeze film lubrication with second order fluid as lubricant between two parallel plates wherein the upper plate approaches the lower plate with a finite velocity. The influence of lubricant rheology on lubrication characteristics has been examined.

2. Mathematical Formulation and Solution

An incompressible homogeneous second order fluid in motion is governed by Coleman and Noll's constitutive equation

$$S = -pI + \phi_0 A_1 + \phi_1 A_2 + \phi_2 A_1^2 \qquad ...(1)$$

where S is Cauchy stress, $-pI$ the spherical stress due to incompressibility, A_1 and A_2 are the first two Rivlin-Ericksen tensors defined by

$$A_1 = \text{grad } V + (\text{grad } V)^T \quad \text{.. (2a)}$$

$$A_2 = \overset{\circ}{A}_1 + A_1 \text{ grad } V + (\text{grad } V)^T A_1 \quad \text{.. (2b)}$$

where V is the fluid velocity, ϕ_0, ϕ_1 and ϕ_2 are the coefficients of viscosity, viscoelasticity, and cross viscosity, respectively. The dot in eqn. 2 (b) denotes the material time derivative. We consider the two-dimensional squeeze film of such a second order fluid formed between two parallel plates where the upper plate is approaching the lower plate with a finite velocity V_0. The origin O is chosen at the centre of the lower plate, the x-axis is chosen along the lower plate and y-axis perpendicular to it. Let (u, v) be the velocity components in x, y directions respectively. Let $h(t)$ be the gap width between the two plates at time t and $2L$ be the length of the plates. Hence $V_0 = \dfrac{dh}{dt}$. The flow in the fluid film is governed by the following equations of continuity and motion[4,5]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{...(3)}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2} + \beta \left[3 \frac{\partial^2 u}{\partial y^2} \frac{\partial u}{\partial x} + 4 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^3 u}{\partial x\, d y^2} + \frac{\partial^2 v}{\partial y^2} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial y^2} + v \frac{\partial^3 u}{\partial y^3} \right] + 2\gamma \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \quad \text{...(4)}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = 2(2\beta + \gamma) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \quad \text{.. (5)}$$

where p is the pressure and

$$\nu = \phi_0/\rho, \; \beta = \phi_1/\rho, \; \gamma = \phi_2/\rho. \quad \text{...(6)}$$

Equations (3), (4) and (5) are subjected to the following boundary conditions.

$$u = 0, v = 0 \text{ at } y = 0 \quad \text{...(7a)}$$

$$u = 0, v = -V_0 \text{ at } y = h \quad \text{...(7b)}$$

$$p = 0 \qquad \text{at } x = \pm L. \quad \text{...(8)}$$

Following Tanner[6], we seek the solution of equations (3) to (5) in the form[7]

$$u(x, y) = m(x) y^2 + n(x) y \quad \text{...(9)}$$

where the functions $m(x)$ and $n(x)$ are to be determined. Integrating the continuity equation (3) after the substitution of the velocity profile given by eqn. (9) and using the boundary conditions (7), we obtain

$$v(x, y) = -\frac{1}{3}\frac{dm}{dx}y^3 - \frac{1}{2}\frac{dn}{dx}y^2 \qquad ...(10)$$

$$m(x) = (-6V_0 x + C)h^{-3} \qquad ...(11)$$

$$n(x) = -(6V_0 x + C)h^{-2} \qquad ...(12)$$

where C is the arbitrary constant to be determined. Using the equations (9) and (10), eqns. (4) and (5) reduce to

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = 2\nu m + 4(2\beta + \gamma)\frac{d}{dx}(m^2)y^2 + 4(2\beta + \gamma)y\frac{d}{dx}(mn)$$

$$+ \frac{(3\beta + 2\gamma)}{2}\frac{d}{dx}(n^2) \qquad ...(13)$$

$$\frac{1}{\rho}\frac{\partial p}{dy} = 4(2\beta + \gamma)m(2my + n). \qquad ...(14)$$

Integration of equations (13) and (14) yields

$$\frac{1}{\rho}p(x, y) = 2\nu\int m dx + 4(2\beta + \gamma)m^2y^2 + 4(2\beta + \gamma)mny$$

$$+ \frac{(3\beta + 2\gamma)}{2}n^2 + D \qquad ...(15)$$

where D is an arbitrary constant to be determined. The average P of the pressure distribution $p(x, y)$ across the film thickness h is given by

$$P = \frac{1}{h}\int_0^h p(x, y)\,dy \qquad ...(16)$$

which, on using eqn. (15), yields

$$\frac{1}{\rho}P = 2\nu\int m dx + \tfrac{2}{3}(2\beta + \gamma)mh(2mh + 3n)$$

$$+ \tfrac{1}{2}(3\beta + 2\gamma)n^2 + D. \qquad ...(17)$$

Using equation (8) the eqn. (16) reduces to

$$P = 0 \text{ at } x = \pm L. \qquad ...(18)$$

After using the equations (11) and (12) in eqn. (17), the constants C and D satisfying the condition (18) are obtained as

$$C = 0$$

$$D = \frac{6}{h^3}\nu V_0 L^2 - \frac{6}{h^4}(\beta + 2\gamma)V_0^2 L^2 \qquad ...(19)$$

and the average presssure distribution P is obtained as

$$\frac{1}{\rho} P = \frac{6\nu V_0}{h^3} (L^2 - x^2) + \frac{6}{h^4} (\beta + 2\gamma) V_0^2 (x^2 - L^2). \quad \ldots(20)$$

The load carrying capacity W is defined as

$$W = \int_{-L}^{L} \int_{0}^{h} p(x, y)\, dx\, dy. \quad \ldots(21)$$

Using equations (16) and (20) in (21), we obtain

$$\frac{W}{\rho} = \frac{8V_0 L^3}{h^2} \left[\nu - \frac{V_0}{h} (\beta + 2\gamma) \right]. \quad \ldots(22)$$

Solving the equation (22) for V_0, the time-height relation for a constant applied load W is obtained in the form

$$T = -2 \int_{h(t_0)}^{h(t)} \frac{f_1(h)\, dh}{f_2(h) + \sqrt{f_2^2(h) - 4W f_1(h)}} \quad \ldots(23)$$

where

$$f_1(h) = \frac{8L^3}{h^3} (\phi_1 + 2\phi_2)$$

$$f_2(h) = \frac{8\phi_0 L^3}{h^2}$$

$$T = t - t_0$$

and t_0 is the reference time. The non-dimensional average pressure distribution $\bar{P}$, load carrying capacity $\bar{W}$ and time of approach $\bar{T}$ can be obtained as

$$\bar{P} = \frac{P h^2}{\phi_0 V_0 L} = \frac{6}{\epsilon} (1 - \bar{x}^2)(1 - N_n) \quad \ldots(24)$$

$$\bar{W} = \frac{W h}{\phi_0 V_0 L^2} = \frac{8}{\epsilon} (1 - N_n) \quad \ldots(25)$$

$$\bar{T} = \frac{\phi_0 T}{(\phi_1 + 2\phi_2)} = -\log \left[\frac{1 - (1 - \frac{1}{2} B_n H)^{1/2})}{1 + (1 - \frac{1}{2} B_n H^{1/2})} \right] + \frac{2}{1 + (1 - \frac{1}{2} B_n H)^{1/2}} + \log \left[\frac{1 - (1 - \frac{1}{2} B_n)^{1/2}}{1 + (1 - \frac{1}{2} B_n)^{1/2}} \right] \frac{2}{1 + (1 - \frac{1}{2} B_n)^{1/2}} \quad \ldots(26)$$

where

$$\bar{x} = x/L$$

$$\epsilon = h/L$$

$$N_n = \frac{(\phi_1 + 2\phi_2)\, V_0}{\phi_0\, h} \qquad \ldots(27)$$

$$H = h\,(t)/h\,(t_0)$$

$$B_n = \frac{(\phi_1 + 2\phi_2)\, W\, h\,(t^0)}{\phi_0^2\, L^3}.$$

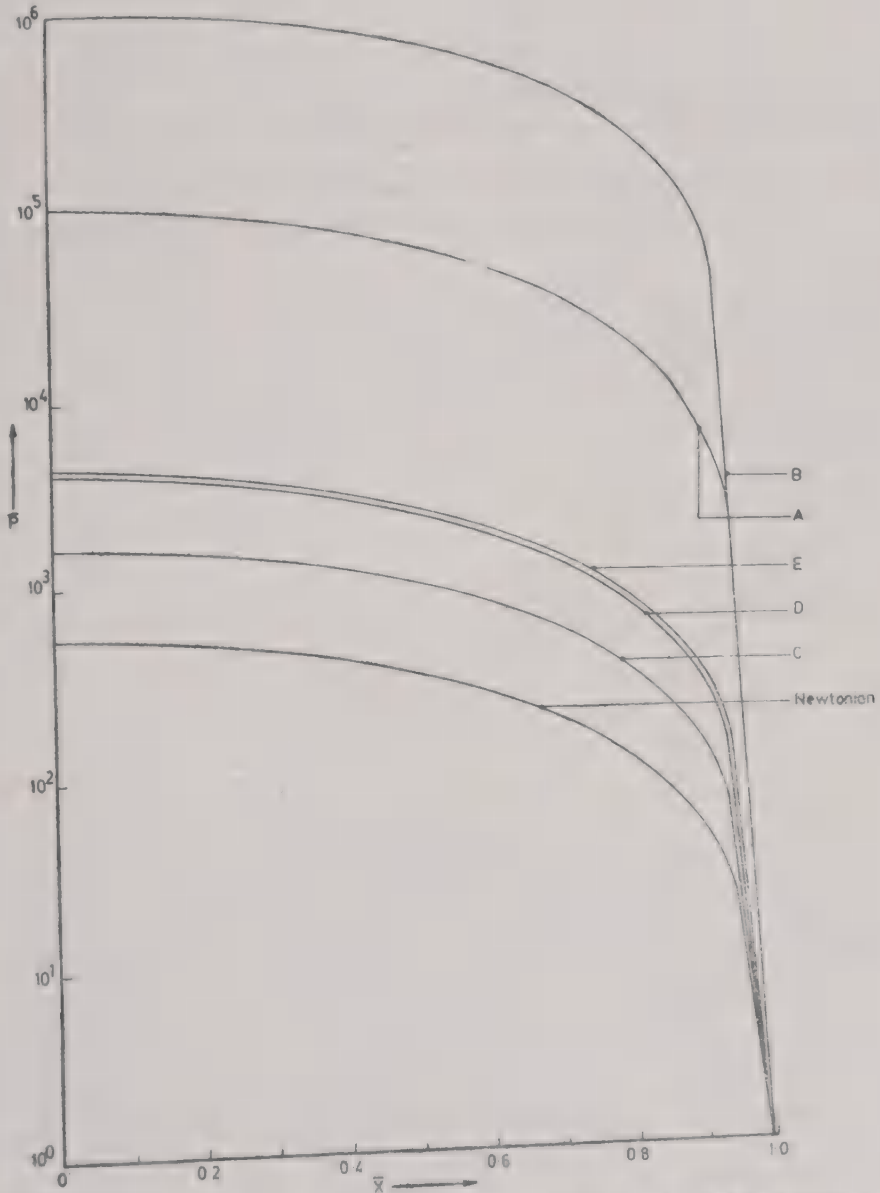

FIG. 1. The dimensionless axial pressure distribution for various fluid samples $A - E$ ($L = 1.0$; $h = 0.01$, $V_0 = -0.5$).

3. Discussion and Conclusion

For the purpose of numerical calculation, we have taken the values of the material parameters from references Lai *et al.*[8]. and Markovitz[9], which are given in Table I.

TABLE I

Fluid	Description	ϕ_0 (p)	ϕ_1 (ps)	ϕ_2 (ps)
A	Normal old human synovial fluid[8]	21.6	—24.1	48.2
B	Normal young human synovial fluid[8]	82	—975	1950
C	Osteoarthritic fluid[8]	2.5	—0.025	0.05
D	Polyisobutylene in cetane[9] 5.4% at 30°C	18.5	—0.2	1.0
E	Polyisobutylene in cetane[9] 5.39% at 30°C	18.5	—0.32	1.12

The Fig. 1 shows the axial pressure distribution. The pressure build up for second order fluids is found to be considerably higher than that for the Newtonian fluid of the

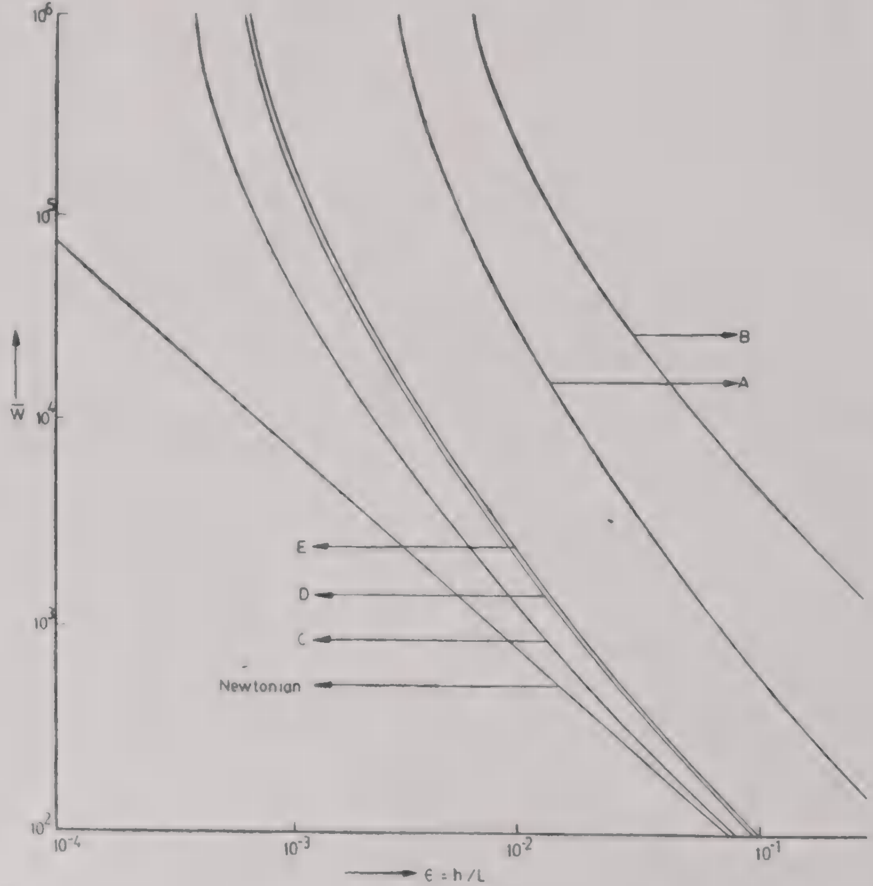

FIG. 2. The dimensionless load carrying capacity $\overline{W}$ *vs* ϵ for various fluid samples *A*—*E* ($L = 1.0$, $V_0 = -0.5$).

the same viscosity which can be distributed to the normal stress effects. The graph of load carrying capacity $\overline{W}$ versus ϵ is given in the Fig. 2. The load carrying capacity for second order fluids is significantly higher than that for the Newtonian case. Also, from the Fig. 3, we observe that the time of approach for second order fluids is found to be delayed in comparison with that for the Newtonian fluid. From Figs 1—3, it is observed that fluid samples with larger numerical values of ϕ_1, ϕ_2 yield higher load capacity and delayed time of approach compared with smaller values. This implies that the possible contact of lubricating surfaces is delayed for a longer time in case of second order fluid lubricant. Thus, the second-order fluids behave as effective lubricants.

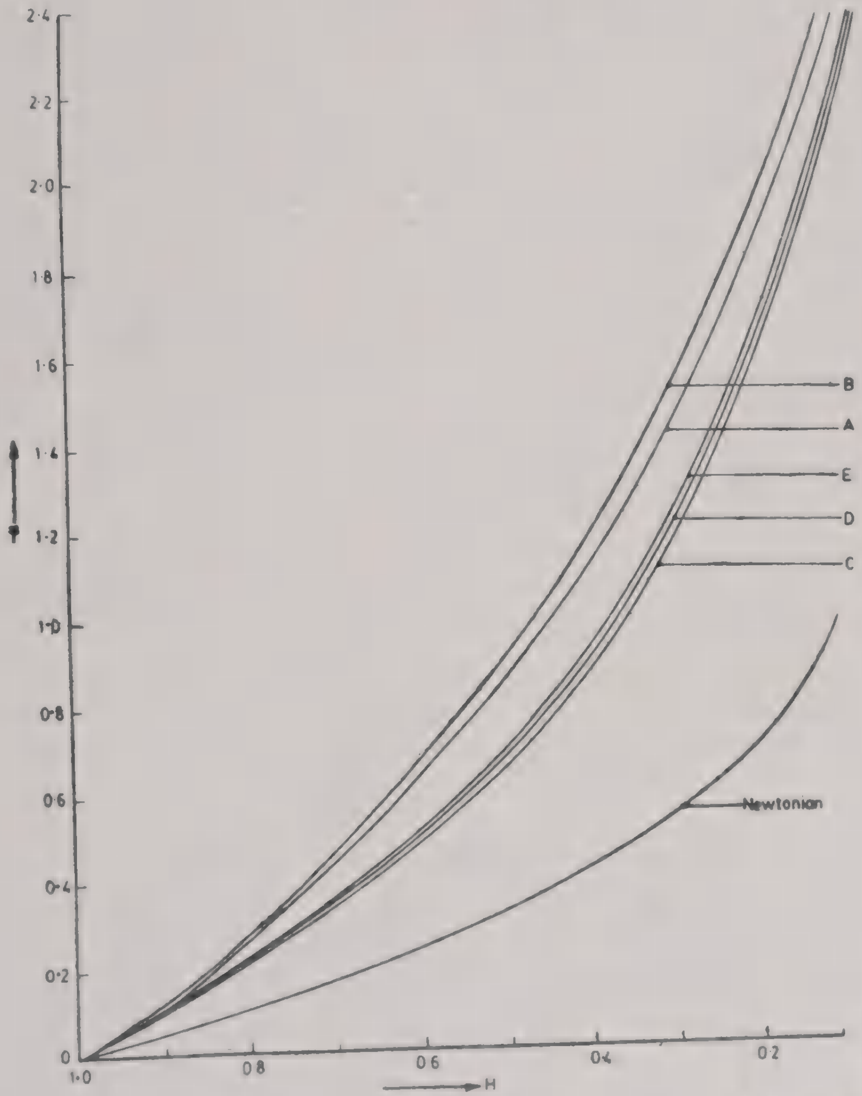

FIG. 3. The time.height relation for varions fluid samples $A - E$ $(L = 1.0;\ h_0 = 0.1;\ W = 100)$.

ACKNOWLEDGEMENT

The authors are grateful to the referee for helpful remarks.

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CIRCULAR ORBITS OF CHARGED TEST PARTICLES IN RIESSNER-NORDSTROM FIELD

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Criteria for the existence and stability of circular orbits of a charged particle in Reissner-Nordstrom field have been discussed using the path deviation equations. Vibration frequencies in the orbital plane and the plane perpendicular to it are determined and the Shirokov effect of the Einsteinian theory of gravitation is obtained in the presence of charges.

1. INTRODUCTION

In this paper we have discussed the existence and stability of circular orbits of a charged test particle in the Nordstrom field using the path deviation equations. In the absence of the central charge the discussion reduces to that of the motion of a particle in the Schwarzschild field and we get back the effect of the Einsteinian theory of gravitation obtained by Shirokov[1]. Even in the case of the motion of a neutral test particle in the field of the charged central body we get the additional gravitational effects due to the central charge obtained by Howes[2] for the Nordstrom metric in the absence of the cosmological constant.

A comparison of the two results, namely, the results of this paper and the one obtained by Shirokov reveals the additional graviational effects which are called into play due to the presence of the charges.

The equations of motion of charged test particle of mass m_0 and charge e_0 are given by generalized Lorentz equations

$$u^i_{;j}\, u^j = -\frac{e_0}{m_0} F^i_j \; u^j \qquad \text{...(1.1)}$$

where u^i ($= dx^i/ds$) is the unit 4-velocity of the charged test particle and a semi-colon (;) denotes covariant differentiation. The path being given by eqns (1.1) the deviation equations with the help of the results given in Synge[3] are obtained in the form

$$\frac{d^2\xi^i}{ds^2} + 2\,\Gamma^i_{jk}\,u^j\left(\frac{d\xi^k}{ds}\right) + \frac{\partial}{\partial x^l}\left(\Gamma^i_{jh}\right)u^j\,u^k\,\xi^l$$

$$= -\frac{e_0}{m_0}\left\{F^i_j\left(\frac{d\xi^j}{ds}\right) + 2\,F^i_j\;\Gamma^j_{mk}\,\xi^m\,u^k + \frac{\partial}{\partial x^n}\left(F^i_j\right)\xi^n\,u^j\right\} \qquad \ldots(1.2)$$

where ξ^i is the small four vector whieh gives the deviation of the test particle from the fiducial path

2. Circular Orbits in the Nordstrom Field

The gravitational field of a central charged body is described by the Reissner-Nordstrom metric

$$ds^2 = -\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right)^{-1} d\gamma^2 - \gamma^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

$$+\left(1 - \frac{2m}{\gamma^2} + \frac{\epsilon^2}{\gamma^2}\right)dt^2. \qquad \ldots(2.1)$$

The only surviving component of the eletromagnetic field tensor F_{ij} is $F_{41} = \epsilon/\gamma^2$. The arbitrary constants of integration m and ϵ appearing in eqn. (2.1) are identified as mass and charge of the gravitating body respectively.

In order to determine the expressions for basic velocities in the Nordstrom field we suppose that the fiducial path is a circular trajectory with γ = constant in the plane $\theta = \pi/2$. This gives $u^1 = d\cdot/ds = 0$ and $u^2 = d\theta/ds = 0$; and the only non-vanishing components of the velocity are $u^3 = d\phi/ds$ and $u^4 = dt/ds$. From equations of motion (1.1) and the line-element (2.1) we get respectively

$$(u^3)^2 = \left(\frac{m}{\gamma^3} - \frac{\epsilon^2}{\gamma^4}\right)^2 (u^4) + \frac{e_0}{m_0}\left(\frac{\epsilon}{\gamma^3}\right)u^4 \qquad \ldots(2.2)$$

and

$$(u^3)^2 = \frac{\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right)(u^4)^2 - 1}{\gamma^2}. \qquad \ldots(2.3)$$

From these equations the explicit expressions for the components u^3 and u^4 are obtained as

$$u^3 = \frac{\left[\left(\frac{2m}{\gamma} - \frac{2\epsilon^2}{\gamma^2}\right)\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) \times \left\{\left(\frac{\epsilon_0}{m_0}\,\frac{\epsilon}{\gamma}\right)^2 \pm \frac{e_0}{m_0}\,\frac{\epsilon}{\gamma}\right.\right.}{\sqrt{2}\,\gamma\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right)}$$

(equation continued on p. 643)

$$\times \left(4 \left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \frac{e_0^2}{m_0^2} \frac{\epsilon^2}{\gamma^2}\right)^{1/2}\Bigg\}\Bigg] \qquad \ldots(2.4)$$

and

$$u^4 = \frac{\frac{e_0}{m_0} \frac{\epsilon}{\gamma} \pm \left(4\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \frac{e_0^2}{m_0^2} \frac{\epsilon^2}{\gamma^2}\right)^{1/2}}{2\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right)}. \qquad \ldots(2.5)$$

For the circular orbits to exist u^3 and n^4 must be real i. e.

$$(u^3)^2 > 0 \text{ and } (u^4)^2 > 0 \qquad \ldots(2.6)$$

and this determines the existence region. Which is same as obtained by Howes[2] in the absence of the cosmological constant for aneutral test particle. Therefore we may conclude that charge of the particle does not have any effeet on existence region.

To discuss the stability of the circular orbits we impart a small momentum distrubance $d\xi^i/ds$ to the orbiting test particle. If stable, its consequent vibrations may be in the orbital plane and also in the plane perpendicular to it. Since θ disturbances are independent of γ, ϕ, t perturbations we consider equations (1.2) with $i = 2$ in order to discuss the stability in the plane perpendicular to the orbital plane.
This gives

$$\frac{d^2 \xi^2}{ds^2} + (u^3)^2 \xi^2 = 0. \qquad \ldots(2.7)$$

The solution of this equation is

$$\xi^2 = \xi_0^2 \, e^{i \alpha s} \qquad \ldots(2.8)$$

where

$$\Omega^9 = (u^3)^2. \qquad \ldots(2.9)$$

In the region where the circular orbits, will be stable to disturbances perpendicular to orbital plane for real Ω which is the situation in view of eqns. (2.6) and (2.9). The periods of these vibuations will be $2\pi/\Omega$.

To examine stability in the orbital plane we consider eqns. (1.2) with $i = 1, 3, 4$ and obtain,

$$\frac{d^2 \xi^1}{ds^2} + a_1 \frac{d\xi^3}{ds} + a_2 \frac{d\xi^4}{ds} + a_3 \xi' = 0 \qquad \ldots(2.10)$$

$$\frac{d^2 \xi^3}{ds^2} + b \frac{d\xi^1}{ds} = 0 \qquad \ldots(2.11)$$

and

$$\frac{d^2 \xi^4}{ds^2} + C_1 \frac{d\xi^1}{ds} + C_2 \xi^3 + C_3 \xi^4 = 0 \qquad \text{...(2.12)}$$

where

$$a_1 = -2\gamma \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) u^3 \qquad \text{...(2 13)}$$

$$a_2 = \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) \left\{\left(\frac{2m}{\gamma^2} - \frac{2\epsilon^2}{\gamma^3}\right) u^4 + \frac{e_0}{m_0} \frac{\epsilon}{\gamma^2}\right\} \qquad \text{...(2.14)}$$

$$a_3 = \left(-1 + \frac{\epsilon^2}{\gamma^2}\right)(u^3)^2 + \left\{\frac{6m^2}{\gamma^4} + \frac{5\epsilon^4}{\gamma^6} - \frac{12m\,\epsilon^2}{\gamma^5} - \frac{2m}{\gamma^3} + \frac{3\,\epsilon^2}{\gamma^4}\right\} \times (u^4)^2 + \left\{\frac{6m}{\gamma^4} - \frac{4\epsilon^3}{\gamma^5} - \frac{2\epsilon}{\gamma^3} + \frac{2e_0}{m_0} \times \left(\frac{m}{\gamma^2} - \frac{\epsilon^2}{\gamma^3}\right)\right\} u^4 \qquad \text{...(2.15)}$$

$$b = \frac{2}{\gamma} u_3 \qquad \text{...(2.16)}$$

$$c_1 = \frac{1}{\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^3}\right)} \left\{\left(\frac{2m}{\gamma^2} - \frac{2\epsilon^2}{\gamma^3}\right) u^4 + \frac{c_0}{m_0} \frac{\epsilon}{\gamma^2}\right\} \qquad \text{...(2.17)}$$

$$c_2 = -\frac{2e_0}{m_0} \frac{\epsilon}{\gamma} u_3 \qquad \text{...(2.18)}$$

$$c_3 = \frac{2e_0}{m_0} \frac{\epsilon}{\gamma^2} \left(\frac{m}{\gamma^2} - \frac{\epsilon^2}{\gamma^3}\right) u^4. \qquad \text{...(2.19)}$$

In order to solve eqns. (2.10) — (2.12) we make use of the substitutions

$$\xi^1 = \xi^1_0\, e^{i\omega s};\ \xi^3 = \xi^3_0\, e^{i\omega s} \text{ and } \xi^4 - \xi^4_0\, e^{i\omega s} \qquad \text{...(2.20)}$$

is eqns. (2.10) - (2.12) and obtain

$$(a_3 - \omega^2)\, \xi^1_0 + a_1 (i\,\omega)\, \xi^3_0 + a_2 (i\omega)\, \xi^4_0 = 0 \qquad \text{...(2.21)}$$

$$b\,(i\omega)\, \xi^1_0 + (-\omega^2)\, \xi^3_0 = 0 \qquad \text{...(2.22)}$$

and

$$c_1\,(i\omega)\, \xi^1_0 + c_2\, \xi^3_0 + (c_3 - \omega^2)\, \xi^4_0 = 0. \qquad \text{...(2,23)}$$

For these equations to yield non-trivial solutions we require

$$\begin{vmatrix} (a_3 - \omega^2) & a_1 i\omega & a_2 i\omega \\ bi\omega & -\omega^2 & 0 \\ c_1 i\omega & c_2 & (c_3 - \omega^2) \end{vmatrix} = 0 \qquad \text{...(2.24)}$$

This gives on simplification the square of the vibration frequency in the orbital plane as $\omega^2 = 0$ and

$$\omega^2 = \frac{-(a_2 c_1 + a_1 b - a_2 - c_3) \pm [(a_2 c_1 + a_1 b - a_3 - c_3)^2 - 4(a_3 c_3 + a_2 bc_2 - a_1 bc_3)]^{1/2}}{2} \qquad \text{..(2.25)}$$

where a_1, a_2, a_3, b, c_1, c_2 and c_3 are given by equations (2.13) – (2.19). Stability to disturbances in the orbital plane requires $\omega^2 > 0$ and the period of these vibrations is given by $2\pi/\omega$. The Shirokov general relativistic effect in the presence of charges is given by the difference in the periods of vibrations namely, by $2\pi/\Omega - 2\pi/\omega$ where Ω and ω are given by eqns. (2.9) and (2.25) respectively. Although the existence region of the circular orbits and the stabilty to the disturbances prependicular to the orbital plane are independent of the terms involving e_0/m_0 but the stability to disturbances in the orbital plane is effected by terms involving e_0/m_0. However it is difficult to express the ettect of such terms explicitly. If we take $e_0 = 0$ in the above equations we get the results obtained by Howes[2] for a neutral test particle in the Nordstrom field with $\lambda = 0$ and if we put $\epsilon = 0$ in the above equations we get the results obtained by Shirokovo[1].

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RANDOM RAYLEIGH WAVES IN NON-HOMOGENEOUS ELASTIC MEDIA

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The problem of propagation of Rayleigh waves in a semi-infinite elastic medium having a vertical non-homogeneity has been considered following Beltzer[1]. The variances of the displacements and velocities for a stationary white noise at a point on the boundary has been numerically evaluated.

1. INTRODUCTION

Earthquakes and random material defects give rise to Rayleigh waves of strengths randomly varying with frequency. Beltzer[1] considered such waves for elastic and viscoelastic media. He assumed the medium to be homogeneous. It is natural therefore to ask how the response will be modified for a non-homogeneity of known type. We have investigated here the problem of random waves generated in elastic non-homogeneous medium. In order to make the problem tractable the vertical non-homogeneity is assumed to be of a type admitting decoupling of the equations of motion. The variance of the displacements and velocities are obtained and are compared with the values obtained by Beltzer[1].

2. RAYLEIGH WAVES IN NON-HOMOGENEOUS ISOTROPIC MEDIUM

Let us consider the propagation of a plane wave through an isotropic elastic non-homogeneous half-space with a free plane boundary.

For simplicity we take the boundary as $z = z_1$ with positive z towards the interior of the solid and take the plane wave travelling in the x-direction. Stoneley [see Ewing p. 350] took the equations for two dimensional motion as

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda\theta + 2\mu\frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right\} \quad \text{...(1)}$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left(\lambda\theta + 2\mu\frac{\partial u_z}{\partial z}\right). \quad \text{...(2)}$$

Taking $\lambda = \mu =$ a function of z only and ρ is also a function of z only,

$$\text{put } u_x = u'_x/\sqrt{\mu} \quad \text{and} \quad u_z = u'_z/\sqrt{\mu} \quad \text{...(3)}$$

then the eqns. (1) and (2) become

$$\frac{\rho}{\sqrt{\mu}}\frac{\partial^2 u_x'}{\partial t^2} = 3\sqrt{\mu}\frac{\partial^2 u_x'}{\partial x^2} + 2\sqrt{\mu}\frac{\partial^2 u_z'}{\partial z\,\partial x} + \sqrt{\mu}\frac{\partial^2 u_x'}{\partial z^2} + \frac{u_x'}{4\mu^{3/2}}\left(\frac{d\mu}{dz}\right)^2 - \frac{u_x'}{2\sqrt{\mu}}\frac{d^2\mu}{dz^2} \qquad \ldots(4)$$

$$\frac{\rho}{\sqrt{\mu}}\frac{\partial^2 u_z'}{\partial t^2} = \sqrt{\mu}\frac{\partial^2 u_z'}{\partial x^2} + 2\sqrt{\mu}\frac{\partial^2 u_x'}{\partial z\,\partial x} + 3\sqrt{\mu}\frac{\partial^2 u_z'}{\partial z^2} + \frac{3u_z'}{4\mu^{3/2}}\left(\frac{d\mu}{dz}\right)^2 - \frac{3u_z'}{2\sqrt{\mu}}\frac{d^2\mu}{dz^2}. \qquad \ldots(5)$$

Taking $\mu = \mu_0 (z/z_0)^2$ and $\rho = \rho_0 (z/z_0)^2$ ...(6)

where μ_0, ρ_0 are constants, then (4) and (5) reduce to

$$\rho_0\frac{\partial^2 u_x'}{\partial t^2} = \mu_0\left[3\frac{\partial^2 u_x'}{\partial x^2} + 2\frac{\partial^2 u_z'}{\partial z\,\partial x} + \frac{\partial^2 u_x'}{\partial z^2}\right] \qquad \ldots(7)$$

$$\rho_0\frac{\partial^2 u_z'}{\partial t^2} = \mu_0\left[\frac{\partial^2 u_z'}{\partial x^2} + 2\frac{\partial^2 u_x'}{\partial z\,\partial x} + 3\frac{\partial^2 u_z'}{\partial z^2}\right]. \qquad \ldots(8)$$

Taking

$$u_x' = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}\ ;\ u_z' = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}. \qquad \ldots(9)$$

ϕ, ψ being functions of x, z and t only, then, from (7) and (8), we get

$$\alpha_0^2\,\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} \qquad \ldots(10)$$

$$\beta_0^2\,\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2} \qquad \ldots(11)$$

where

$$\alpha_0^2 = \frac{3\mu_0}{\rho_0}\ ;\quad \beta_0^2 = \frac{\mu_0}{\rho_0} \qquad \ldots(12)$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

Assuming a wave travelling in the x-direction of wave length $2\pi/k$,

We get

$$\phi = g(z) \exp(i k (x - ct))$$
$$\psi = h(z) \exp(i k (x - ct)). \qquad \ldots(13)$$

Substituting (13) in (10) and (11) we get

$$g(z) = A_1 e^{-rz} \text{ and } h(z) = A_2 e^{-sz}$$

where

$$\left.\begin{aligned} r^2 &= k^2\left(1 - \frac{c^2}{\alpha_0^2}\right); \; c < \alpha_0 \\ s^2 &= k^2\left(1 - \frac{c^2}{\beta_0^2}\right); \; c < \beta_0 \end{aligned}\right\} \qquad \ldots(14)$$

$\therefore$
$$\phi = A_1 e^{-rz} \exp(i k (x - ct)) \qquad \ldots(15)$$
$$\psi = A_2 e^{-rz} \exp(i k (x - ct)). \qquad \ldots(16)$$

The boundary conditions

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \quad \text{on } z = z_1, \text{ give}$$

$$\left.\begin{aligned} &A_1 i k e^{-rz_1} (2rz_1 + 1) + A_2 e^{-s_1} [(k^2 + s^2) z_1 + s] = 0 \\ &A_1 e^{-rz_1} [z_1 (3r^2 - k^2) + 3r] - A_2 i k e^{-sz_1} (2z_1 s + 3) = 0 \end{aligned}\right\}. \qquad \ldots(17)$$

Eliminating A_1 and A_2 we get the frequency equation

$$z_1^2 \mid k \mid^2 \left[4 \sqrt{\left(1 - \frac{c^2}{\beta_0^2}\right)\left(1 - \frac{c^2}{3\beta_0^2}\right)} - \left(2 - \frac{c^2}{\beta_0^2}\right)^2 \right]$$
$$+ z_1 \mid k \mid \frac{c^2}{\beta_0^2} \left(3 \sqrt{1 - \frac{c^2}{3\beta_0^2}} + \sqrt{1 - \frac{c^2}{\beta_0^2}} \right)$$
$$+ 3 - 3 \sqrt{\left(1 - \frac{c^2}{\beta_0^2}\right)\left(1 - \frac{c^2}{3\beta_0^2}\right)} = 0. \qquad \ldots(18)$$

The above equation is consistent with values of c/β_0 greater than a fixed number ξ (the fixed number ξ represents the Rayleigh wave velocity in a homogeneous medium with β_0 as S wave velocity). Hence the possible Rayleigh waves in the medium under consideration have velocities ranging from ξ to 1, and the corresponding frequency from ω_0 to ∞ (ω_0 is the frequency corresponding to wave velocity ξ). We shall consider random Rayleigh waves with frequencies lying in the above range.

Now we can write the displacement u_x and u_z in the following way

$$\left.\begin{aligned} u_x &= BF_x(x, z)\, e^{-ikct} \\ u_z &= BF_z(x, z)\, e^{-ikct} \end{aligned}\right\} \qquad \ldots(19)$$

where

$$\left.\begin{aligned} F_x &= \dot{z}\, k \sqrt{\mu}\ \frac{z}{z_1} \left[e^{-rz} - \frac{s\,(2rz_1 + 1)}{z_1\,(k^2 + s^2) + s} . e^{-rz_1} . e^{s(z_1 - z)} \right] e^{ikx} \\ F_z &= \sqrt{\mu}\ \frac{z}{z_1} \left[- re^{-rz} + \frac{k^2\,(2rz_1 + 1)}{z_1\,(k^2 + s^2) + s} . e^{-rz_1} . e^{s(z_1 - z)} \right] e^{ikx} \end{aligned}\right\} . \qquad \ldots(20)$$

3. Basic Equations in the Random Case

Making use of the arbitrary nature of the value B we can introduce a random complex process $B(\omega)$, $\omega = kc$, with zero mean and uncorrelated increments such that for any interval (ω_1, ω_2) (Beltzer[1]).

$$< \mid B(\omega_2) - B(\omega_1) \mid^2 > = \int_{\omega_1}^{\omega_2} S_B(\omega)\, d\omega \qquad \ldots(21)$$

where $< >$ denotes averaging and S_B is the spectral density. The stationary random Rayleigh waves are now defined as stochastic integrals which describe the superposition of the waves given by (19)

$$u_i(x, z, t) = \int_{-\infty}^{\infty} F_i(\omega, x, z)\, e^{-i\omega t}\, dB(\omega), \ (i = x, z) \qquad \ldots(22)$$

where F_i are given in (20).

The waves defined by (22) can be described also in the form of stochastic integrals

$$u_i(x, z, t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, dE_i(\omega, x, t), \ (i = x, z) \qquad \ldots(23)$$

where $E_i(\omega, x, z)$ describes complex amplitudes at apoint (x, z) for random processes with zero mean and uncorrelated increments.

By comparison between (22), (23) we get

$$E_i(\omega, x, z) = \int_{-\infty}^{\omega} F_i(\delta, x, z)\, dB(\delta), \ (-\infty < \omega < \infty) \qquad \ldots(24)$$

and the spectra s_E^i are given by

$$S_E^i(\omega, x, z) = S_B(\omega) \mid F_i(\omega, x, z) \mid^2, \ (\dot{z} = x, z). \qquad \ldots(25)$$

We consider the random motion at the point $(0, z_1)$ to be prescribed. We get the complex random amplitudes and the spectra during propagation from (24), (25) as

$$E_i(\omega, x, z) = \int_{-\infty}^{\infty} f_i(\delta, x, z)\, dE_i(\delta, 0, z_1) \qquad \ldots(26)$$

and

$$S_E^i(\omega, x, t) = S_E^i(\omega, 0, z_1) \, | f_i(\omega, x, z) |^2 \qquad \ldots(27)$$

where $E_i(\omega, 0, z_1)$ and $S_E^i(\omega, 0, z_1)$ are the amplitudes and the spectra at $(0, z_1)$ and

$$f_i(\omega, x, z) = F_i(\omega, x, z)/F_i(\omega, 0, z_1), \; (i = x, z). \qquad \ldots(28)$$

The processes $u_i(x, z, t)$ are not independent and (24), (25) yield the coupling equations between their random amplitudes and their spectra :

$$E_x(\omega, x, z) = \int_{-\infty}^{\infty} \psi_{xz}(\delta, z)\, dE_z(\delta, x, z) \qquad \ldots(29)$$

and

$$S_E^x(\omega, x, z) = S_E^z(\omega, x, z) \, | \psi_{xz}(\omega, z) |^2 \qquad \ldots(30)$$

where

$$\psi_{xz}(\omega, z) = F_x(\omega, x, z)/F_z(\omega, x, z). \qquad \ldots(31)$$

4. Random Rayleigh Waves in Non-homogeneous Elastic Media

The functions $f_i(\omega, x, z)$ and $\psi_{xz}(\omega, x, z)$, which govern the evaluation of the processes $u_i(x, z, t)$ and the coupling between them can be written according to eqns. (20), (28), (31) in the following way :

$$f_x(\omega, x, z) = \frac{z}{z_1}\left[\frac{e^{r(z_1-z)} - Pse^{s(z_1-z)}}{1 - Ps}\right] e^{ikx} \qquad \ldots(32)$$

$$f_z(\omega, x, z) = \frac{z}{z_1}\left[\frac{-re^{r(z_1-z)} + Pk^2 e^{s(z_1-z)}}{-r + Pk^2}\right] e^{ikx} \qquad \ldots(33)$$

$$\psi_{xz}(\omega, z) = ik\left[\frac{e^{r(z_1-z)} - Pse^{s(z_1-z)}}{-re^{r(z_1-z)} + rk^2 e^{s(z_1-z)}}\right] \qquad \ldots(34)$$

where

$$P = \frac{2rz_1 + 1}{z_1(k^2+s^2) + s} \qquad \ldots(35)$$

and r, s are given in (14).

Let $\bar{S}$ be the intensity of a white noise disturbance in the x-direction at $r = (0, z_1)$:

$$S_E^x\ (\omega, 0, z_1) = \bar{S};\ |\ \omega\ | < \infty. \qquad \text{...(36)}$$

Then from (31) & (34) we get

$$S_E^z(\omega, 0, z_1) = \frac{\bar{S}\,(-r + Pk^2)}{k^2\,(1 - Ps)^2}\ . \qquad \text{...(37)}$$

Then we get, the variances of the displacements and their nth derivatives as

$$\text{Var}\ [u_i^{(n)}\ (x, z)] = \int_{-\infty}^{\infty} \omega^{2n}\ S_E^i\ (\omega, x, z)\ d\omega$$

$$= \int_{-\infty}^{\infty} \omega^{2n}\ |\ f_i\ (\omega, x,\ z)\ |^{\ 2}\ S_E^i\ (\omega, o, z_1)\ d\omega. \qquad \text{...(38)}$$

So, we have the result :

$$\text{Var}\ [u_x^{(n)}\,(x, z)] = \int_{-\infty}^{\infty} \omega^{2n} \left[\frac{z}{z_1}\ \frac{e^{r(z_1 - z)} - Pse^{s(z_1 - z)}}{1 - Ps} \right]^2 \bar{S}\ d\omega \qquad \text{...(39)}$$

and

$$\text{Var}\ [u_z^{(n)}\ (x, z)] = \int_{-\infty}^{\infty} \omega^{2n} \left[\frac{z}{z_1}\ \frac{-re^{r(z_1 - z)} + Pk^2\ e^{s(z_1 - z)}}{k\ (1 - Ps)} \right]^2 \bar{S}\ d\omega \qquad \text{...(40)}$$

where P, r, s, are given in (35), (14).

5. Discussion

The variance of u_x, u_z and $\frac{du_x}{dt}$, $\frac{du_z}{dt}$ have been plotted in Fig. 1 and Fig. 2 respectively against z/z_1. Curves of the form var (u_x) = constant$/z^n$, and similarly for others, have been fitted to the data in order to compare them with the homogeneous case of Beltzer[1]. The values of n obtained are as follows :

n	u_x	u_z	$\frac{du_x}{dt}$	$\frac{du_z}{dt}$
Homogeneous case (actual value)	1.0	1.0	3.0	3.0
Non homogeneous case (data based)	0.334	2.106	1.020	2.474

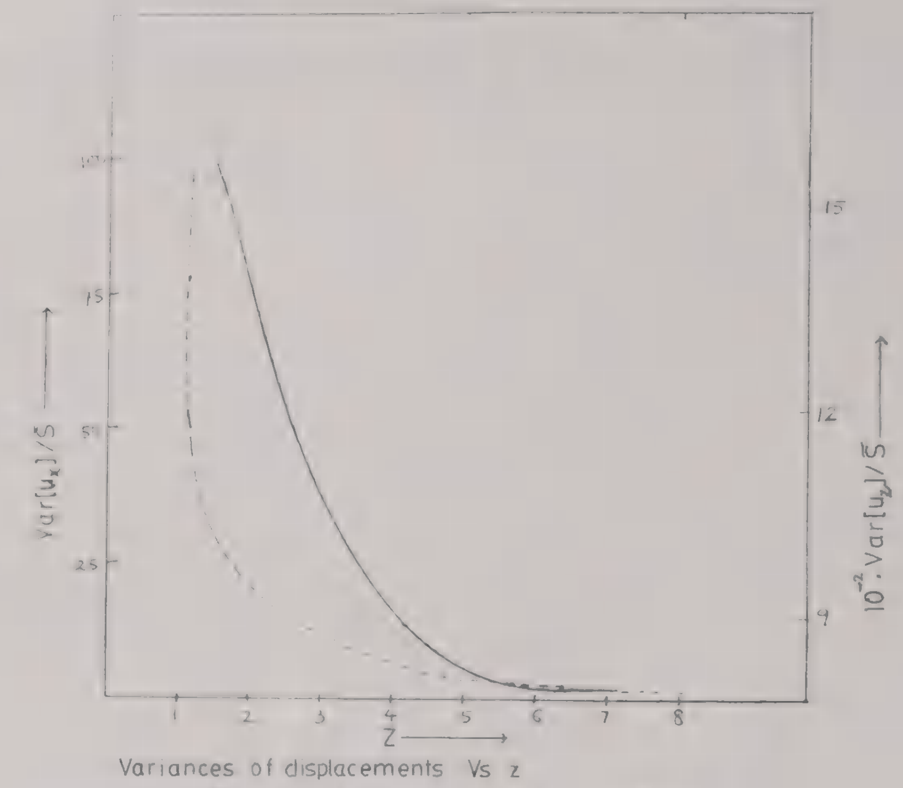

FIG. 1. Variances of displacements *Vs z*.

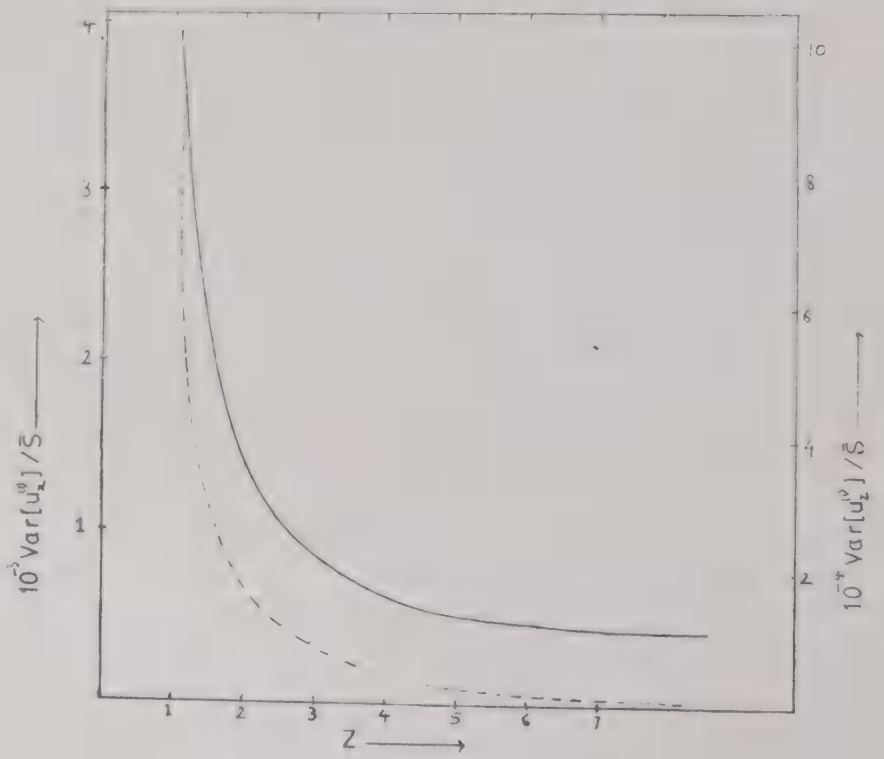

FIG. 2. Variances of velocities *Vs z*.

The vertical non-homogeneity considered in this problem therefore diminishes the change of variance of u_x, $\frac{du_x}{dt}$, $\frac{du_z}{dt}$ with depth compared to the homogeneous case, while var (u_z) increases with depth relative to the homogeneous case.

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